Exploring the role of representations when young children solve a combinatorial task

Hanna Palmér and Jorryt van Bommel

This paper is about the representations young children spontaneously use when they are solving a combinatorial task. The paper describes connections between the representations used by the children and how they solve the combinatorial task, and considers whether the results from studies regarding representations of quantity also apply to combinatorial tasks. Our results indicate some connections between the representations used and the solutions presented, but these connections do not seem to apply to the results from studies of quantity. Some possible explanations for this are outlined in the paper, but more studies will be needed to further elaborate on these issues.

The focus of this paper is on the representations young children spontaneously use when they are solving a (for them) challenging combinatorial task. Most research on young children and mathematics has focused on numbers and quantitative thinking (Sarama & Clements, 2009) and studies on children's representations are often connected to quantity. Thus, there are many studies on young children's representations within the context of quantity but few studies on young children's use of representations when solving tasks within other mathematical areas.

Studies of young children's representations often focus on informal and formal representations. One line of inquiry has looked into linkages and/or development opportunities between informal and formal representations (for example Hughes, 1986; Heddens, 1986; Carruthers & Worthington, 2006). Another line of inquiry has focused on connections between representations used by children and their mathematical abilities (for example Piaget & Inhelder, 1969; Carruthers & Worthington, 2006). In this paper both lines of inquiry will be addressed and the following questions will be elaborated upon:

- Do results from studies of young children's representations of quantity also apply when young children solve combinatorial tasks?

Hanna Palmér, Linnaeus University
Jorryt van Bommel, Karlstad University

- Are there any connections between the representations young children spontaneously use and how they solve a combinatorial task?

Representations

In the paper we deal with the graphic representations young children spontaneously use when solving a combinatorial task. We will not deal with representations of other modalities such as sound, manipulatives or gestures. In the remainder of the paper, then, the word ”representation” will refer only to graphic representations.

Different researchers have divided and identified representations used by children in various ways, and in this section we will present some of these. Then, in the next section, we will connect these to the combinatorial task that will be described in the paper.

> A representation is typically a sign or a configuration of signs, characters or objects. The important thing is that it can stand for (symbolize, depict, encode, or represent) something other than itself.
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> (Goldin & Shteingold, 2001, p. 3).

Children’s drawings are a first step towards using representations since they refer to objects, events, ideas and relationships beyond the surface of the drawing (Piaget & Inhelder, 1969; Matthews, 2006). Children do not distinguish between marks used for writing, mathematics or drawing; they often combine them, and their trials, inventions and combinations are important in developing their understanding of abstract symbolism in mathematics (Carruthers & Worthington, 2006).

Piaget & Inhelder (1969) distinguished between two types of representations used by children: symbols and signs. Symbols include pictures and tally marks that have some resemblance to the objects referred to. Each child can invent such symbols since they are not conventions of society. Signs are conventions of society in the form of spoken and written symbols that do not resemble the objects represented.

As mentioned, studies of how children use representations have often been connected to quantity. One influential study regarding representation of quantity was conducted by Hughes (1986), who investigated how children use their own marks when representing numerals. He identified four forms of marks used by children to represent quantity: idiosyncratic, pictographic, iconic and symbolic. Idiosyncratic marks are irregular representations which cannot be related to the number of objects represented. Pictographic representations are pictures of the represented items, while iconic representations are based on one mark for each item. Symbolic representations are standard forms of representation, for example, numerals and equal signs. The same types of children’s own representations have been identified in later studies, for example by Carruthers and Worthington (2006). Carruthers and Worthington also identified dynamic and written representations.

Representations often have some kind of relation to objects. Heddens (1986) focused on the connection between concrete (objects) and abstract (signs) representations. He introduced pictures and tally marks as two levels between concrete and abstract representations. He referred to representations of real situations, for example, pictures of real items, as semi-concrete, whereas he referred to symbolic representations of concrete items, where the symbols or pictures do not look like the objects they represent, as semi-abstract.

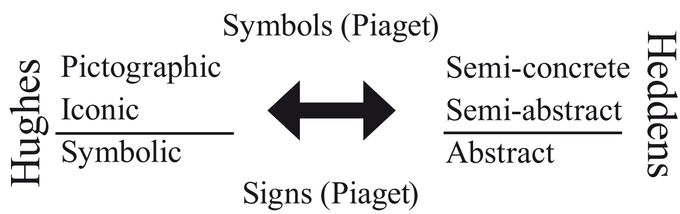

Figure 1. *Connecting Piaget, Hughes and Heddens*

Figure 1 above connects Piaget, Hughes and Heddens. What Hughes named pictographic representations are what Heddens would refer to as semi-concrete, while what Hughes called iconic representations are semi-abstract in Heddens's terms. All of these representations are symbols in Piaget's terms, as each child can invent them. Further, what Hughes named symbolic representations Heddens calls abstract representations and Piaget calls signs. In the remainder of this paper, Hughes's terms will mainly be used when presenting and analysing the results.

Combinatorics

Combinatorics concerns determination of a finite number of discrete structures. These structures offer the opportunity to explore simple combinatorial problems with young children. Combinatorial tasks can also serve as a base on which to build understanding within other areas of mathematics, such as computation, counting and probability. Furthermore, combinatorial tasks facilitate systematic thinking as well as making conjectures and generalizations (English, 2005). The combinatorial task in this study is of the type enumerative combinatorics: counting permutations, in this case for $n=3$.

Research on young children and combinatorics started in the late 1950s with Piaget and Inhelder, who investigated the cognitive development of children's combinatorial and probabilistic thinking. They concluded that children (age 7–8) were not able to work with 2 x 2 or 3 x 3 permutation problems. Later research, however, showed that within a proper and meaningful context, pupils indeed could work effectively with combinatorial situations finding permutations (English, 1991, 2005). The task presented in this paper is embedded in a problem-solving context, which already in the late 1960s was seen as suitable to combine with the practical element within combinatorics (English, 1991).

The major difficulty for young children when solving combinatorial tasks is in listing items systematically (English, 2005). Based on empirical investigations, English (1991) identified five strategies used by young children when working with combinatorial tasks: 1) random selection of objects – with duplicates, 2) trial and error with random item selection – with rejection of duplicates, 3) emerging pattern for the choice of objects – with rejection of duplicates, 4) consistent and complete cyclical item selection – with rejection of duplicates and 5) ”odometer pattern” in item selection. Some of these hierarchical strategies where children start to emerge pattern for the choice of objects are more effective than others when it comes to finding all possible combinations.

The study

The results presented in this paper are from a design research study investigating how to teach mathematics through problem solving in preschool classes. Design research is a cyclic process of designing and testing interventions situated within an educational context (Anderson & Shattuck, 2012). The task in focus in this paper was the third task of six that the children worked on during the intervention. They had already worked on two challenging problem-solving tasks, but not with any combinatorial tasks.

Preschool class

The task was conducted in six preschool classes with a total of 87 children. The classes were selected based on the interest of the teachers at the schools. The Swedish preschool class was implemented in 1998 to facilitate a smooth transition between preschool and primary school and to prepare children for further education. There are no regulations or goals around the teaching of mathematics, but the content of both the preschool and primary school curriculums are to form the basis of the preschool's activities. The working methods and pedagogy are not supposed to be either like school (with a tradition of learning) nor like preschool (with a tradition of play) but a combination of the two (Swedish National Agency for Education, 2014).

The bear task

The combinatorial task on which this paper focuses required children to consider how many different ways three toy bears could be arranged in a row on a sofa. To make the task meaningful for the children it was presented as a conflict between the toy bears, where they could not agree on who should sit at which place on the sofa. One toy bear then suggested that they could change places every day. The task for the children became to find out how many days they could sit in different ways on the sofa.

The children were divided into groups, where approximately 12 children at the time worked on the task. The researchers acted as teachers during the lesson (one researcher per group). When introducing the task, the children were shown three small plastic bears, one red, one yellow and one green. After the introduction the children worked individually. They were given white paper and pencils in different colours but no instructions regarding what or how to do any documentation on the paper. After working alone first for some minutes and then in pairs the children were gathered for a joint discussion based on their documentations. When working in pairs the children compared their documentations to identify similarities and differences. They did not change their documentations. In the joint discussion the different permutations were explored and a joint effort resulted in a display of the combinations with the plastic bears. Finally, the ways the children had documented their solutions were discussed. The purpose of this discussion was to explore the potential of different ways of representing mathematical thinking, to make the children aware of their own and others' use of different marks and to extend the children's repertoire of representations by using peer modelling. Peer modelling implies focusing on children's own marks, discussing ways of representing, meanings and strengths (Carruthers & Worthington, 2006).

Analysing the data

This paper will focus on the influence of the choice of representation, if any, on how the children solved the combinatorial task. The representations referred to are the ones in the children's documentations on paper as described above. These are what Hughes (1986) named pictographic and iconic representations.

When we categorized the documentations as pictographic or iconic representation, we found it necessary to add a category that included both these representation, as some children had used both types. After this, the documentations in each category were categorized once more based on the solution of the task. The number of permutations was a first classification after which the uniqueness also became a component of concern.

Results

As can be seen in table 1, this was a challenging task for the children, and only two of 87 children found six unique permutations when they worked individually with the task. These two children used iconic representation.

Table 1. *Categorization of children's documentation*

Short name	Explanatory statement	Picto-graphic	Picto-graphic & Iconic	Iconic
No new permutations	The child has drawn some toy bears or the combination shown by the teacher and then no further combinations	3		2
Unique permutations A	The child has drawn unique combinations where the total number of combinations is less than six	15 (4)	8 (3)	24 (10)
Unique permutations B	The child has drawn six unique combinations			2
Duplicate permutations	The child has drawn combinations where one or several combinations are duplicated	3		30 (1)
Total		21	8	58

All of the children used some kind of pictographic and/or iconic representation when solving the task; the majority (58 of 87) spontaneously used iconic representation. These iconic representations were of different kinds but always in the colours of the toy bears. Most of these children drew circles or lines, but a few also replaced the toy bears with hearts.

Few children used both pictographic and iconic representations (8 of 87). All of them started with pictographic representation and changed after one or two permutations to some form of iconic representation.

The numbers in parentheses in the category *unique combinations A* represent the number of children who drew three unique combinations and no more, where each toy bear sat at each place once. This indicates some kind of systematization in the solutions as each toy bear is drawn once at each place on the sofa. This was done by 17 children, with four using pictographic representation, three using pictographic and iconic representation, and ten using iconic representation.

None of the children that used pictographic representation had a solution with more than five combinations. This refers both to the children who had unique combinations and those with duplicate combinations. However, the majority of the children that used pictographic representation (15 of 18) produced only unique combinations.

The majority (30 of 33) of the children who made duplicate combinations used iconic representation. Of these, one documentation included the six unique combinations but they were duplicated and thus the child did not seem to recognize the six unique combinations as ”special”.

Discussion and conclusions

In this final section we will focus on the two questions raised in the introduction to the paper:

- Do results from studies of young children's representations of quantity also apply when young children solve combinatorial tasks?
- Are there any connections between the representations young children spontaneously use and how they solve a combinatorial task?

The children in this study had not had any formal instruction regarding representations and, as mentioned, peer modelling was used only after the children had worked on the task individually. When working on the task, all children used pictographic and/or iconic representations. Both of these have a resemblance to the objects they represent (Hughes, 1986). The pictographic representations were drawings of the plastic bears in the three colours, and the iconic representations were made in the three colours.

In studies focused on representations of quantity, pictographic and iconic representations are associated with a lower level of development as they reveal children's attention to each object rather than to the total quantity (Sinclair, Siegrist & Sinclair, 1983). However, in this combinatorial task the children needed to pay attention to each object as well as to the relation between the objects, therefore both pictographic and iconic representations were well suited to it (Hughes, 1986).

The use of iconic representations implies a semi-abstract level (Heddens, 1986); it is more abstract than the use of pictographic representations (semi-concrete level) which were used by fewer children in this study. But, the children who used pictographic representations made fewer duplications than the children who used iconic representations. None of the children who used pictographic representations had a solution with more than five combinations, and the majority of them (15 of 18) drew only unique combinations. Why is that?

Maybe it has to do with time, even though there were no time constraints for this task. It takes longer to draw toy bears than to draw iconic representations, thus the children who used pictographic representations (drew toy bears) had more time to think. Further, drawing toy bears can be experienced as more real, and when working on the task the children who drew the bears could be heard saying things like, "Now it is your turn to sit in the middle" and "He has already been in the middle." Iconic representations are easier to draw, which makes the process faster and maybe that is why the solutions with iconic representations contained a lot of duplications, even when the combinations drawn are few. For example, some children drew three combinations with iconic representations, two of which were duplicates.

Thus the time issue and the connection to real toy bears may be why children who used pictographic representations made few duplications.

According to Devlin (2000), differences in how individuals solve what may look like ”the same” mathematical task can be connected to how the task is described and what it pertains to. As mentioned, the bear task was presented as a conflict between the toy bears, where they could not agree on which of them should sit at which place on the sofa. For some children this seems to have made the task familiar. Saying things like ”Now it is your turn to sit in the middle” and ”He has already been in the middle” indicates an interpretation of the task as a real and familiar situation, and this may be why few duplications were made.

According to English (2005), the major difficulty for young children when solving combinatorial tasks is listing items systematically, and yet another explanation is that the main issue in solving this task is not about the representation used (pictographic and/or iconic representations) but about the systematization of the representations. Regarding systematization Piaget distinguished between empirical abstractions and constructive abstractions (Kamii, Kirkland & Lewis, 2001). Empirical abstractions are generated from empirical experiences where the children focus on certain properties of objects (for example, colour) and ignore others (for example, size). Constructive abstractions are generated from mental actions on objects, not from the objects themselves. Distinguishing relationships between objects when solving a combinatorial task is an example of a constructive abstraction. This means that the relationships that the children need to figure out in the combinatorial task are retrieved from mental actions on the objects. Regardless of whether they use pictographic or iconic representations, the children need to mentally list items systematically to keep track of which combinations they have and have not drawn. Such mental actions on objects are time consuming, which again can be connected to the time issue when drawing toy bears instead of iconic representations.

To sum up, there seem to be some connections between the representations young children spontaneously use and how they solve a combinatorial task. However, these connections seem to differ from the results from studies of young children’s representations of quantity. Iconic representations do not generate a higher level of solution of the combinatorial task; quite the opposite, pictographic representations do seem to imply more systematization and less duplication. Some possible explanations for this have been outlined above, but more studies will be needed to further elaborate on these issues.

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