Skrifter från SMDF, Nr. 10

Editors:
Ola Helenius, Arne Engström, Tamsin Meaney, Per Nilsson, Eva Norén, Judy Sayers, Magnus Österholm

Development of Mathematics Teaching: Design, Scale, Effects

Proceedings of MADIF 9

The Ninth Swedish Mathematics Education Research Seminar
Umeå, February 4-5, 2014

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Preface

This volume contains the proceedings of *MADIF 9*, the Ninth Swedish Mathematics Education Research Seminar, held in Umeå, February 4-2, 2014. The MADIF seminars are organised by the Swedish Society for Research in Mathematics Education (SMDF). MADIF aims to enhance the opportunities for discussion of research and exchange of perspectives, amongst junior researchers and between junior and senior researchers in the field. The first seminar took place in January 1999 at Lärarhögskolan in Stockholm and included the constitution of the SMDF. The second meeting was held in Göteborg in January 2000, the third in Norrköping in January 2002, the fourth and fifth in Malmö in January 2004 and 2006, respectively, and the sixth and seventh in Stockholm in January 2008 and 2010, respectively. Like MADIF 9, the eighth meeting was held in Umeå. Printed proceedings of the seminars are available for all but the very first meeting and in 2015 also online versions will be made available both of the present and of previous volumes.

The members of the 2010 programme committee were Arne Engström (Karlstad University), Ola Helenius (National Center for Mathematics Education, chair), Tamsin Meaney (Malmö Högskola), Per Nilsson (Örebro University), Eva Norén (Stockholm University), Judy Sayers (Stockholm University), and Magnus Österholm (Umeå University. The local organiser was Tomas Bergqvist (Umeå University).

The programme of *MADIF 9* included two plenary lectures by invited speakers Koeno Gravemeijer and Beth Herbel-Eisenmann. There were also a plenary panel consisting of Lisa Björklund-Boistrum, Jeremy Hodgen, Darina Jirotkova and John Mason, moderated by Ola Helenius. As before, MADIF works with a format of full 10 page papers and as well as short presentations. This year was the first where the short presentation (24) outnumbered the full papers (15). It will be interesting to see if this trend continues in 2016. As the research seminars have sustained the idea of offering formats for presentation that enhance feedback and exchange, the paper presentations are organised as discussion sessions based on points raised by an invited reactor. The organising committee would like to express its thanks to the following colleagues for their commitment to the task of being reactors: Annica Andersson, Paul Andrews, Jonas Bergman Ärlebäck, Jorryt van Bommel, Gerd Brandell, Johan Häggström, Darina Jirotkova, Cecilia Kilhamn, Ia Kling Sackerud, Johan Lithner, Hanna Palmér, Kerstin Pettersson, Ann-Sofi Röj-Lindberg, Frode Rönning and Hans Thunberg.

This volume comprises summaries of the two plenary addresses, 15 research reports (papers) and abstracts for the 24 short presentations. In a rigorous two-step review process for presentation and publication, all papers were peer-reviewed by at two to four researchers. Short presentation contributions were reviewed by members of the programme committee. Since 2010, the MADIF Proceedings have been designated scientific level 1 in the Norwegian list of authorised publication channels available at http://dbh.nsd.uib.no/kanaler/.
The editors are grateful to the following colleagues for providing reviews:
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The organising committee and the editors would like to express their gratitude to the organisers of *Matematikbiennalen 2014* for financially supporting the seminar. Finally we would like to thank all participants of *MADIF 9* for sustaining their engagement in an intense scholarly activity during the seminar with its tight timetable, and for contributing to an open, positive and friendly atmosphere.

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Design research on local instruction theories in mathematics education

Koeno Gravemeijer
Eindhoven Technical University

Over the last decades, the view, that teachers have to transmit knowledge, has been replaced with the view that students have to construct knowledge while being supported by teachers and textbooks. It is, however, not immediately clear how to guide and support students in such processes in the case of mathematics education. In response to this problem, design research emerged as a method for developing theories that can function as frameworks of reference for teachers.

Mark that the notion that people construct their own knowledge does not offer a pedagogy. For it implies that students will construct their own knowledge whatever form instruction takes. It does, however, point to the question of *what* it is the students construct. Or, what we want them to construct. This brings us to the question: What do we want mathematics to be for our students? Following Freudenthal (1971) we argue that students should experience mathematics “as a human activity”, as the activity of doing mathematics. According to Freudenthal students should be supported in reinventing mathematics, which fits nicely with the constructivist mantra of students constructing their own knowledge. But how to help students invent or construct what you want them to invent/construct?

In answer to this problem, Simon (1995) coined the term, “hypothetical learning trajectory” (HLT), which refers to choosing tasks with an eye on what they might bring about, envision the mental activities of the students, and anticipate how their thinking might help them to develop the mathematical insights you are aiming for. Being hypothetical, the learning trajectory of course has to be put to the test. When the HLT is enacted, one has to observe students, analyze and reflect upon their thinking, and adjust the HLT. Following this line of thought, we have to support teachers by helping them to design HLT’s, not by offering them scripted textbooks. For, if we want students to reinvent mathematics by doing mathematics, teachers have to adapt to how their students reason and help them build on their own thinking. To do so they need a framework of reference to base their HLT’s on. We may offer them such frameworks in the form of “local instruction theories”—and corresponding resources. A local instruction theory consists of theories about both the process of learning a specific topic and the means to support that learning. The goal of the kind of design research I am discussing here is to develop local instruction theories.

Design research typically encompasses of the following three phases.

1. *Preparing for the teaching experiment*; in this phase, the researchers clarify the theoretical intent, the background theories, the starting points of the students, and the instructional goals; and design a conjectured local instruction theory. Here I want to stress the importance of a sound instructional design theory, as the quality of the research highly depends on the design. The theory of realistic mathematics education that grew out of Freudenthal's adagio of mathematics as a human activity qualifies as such a theory.
2. *Conducting the teaching experiment;* during the teaching experiment the researchers design and adjust instructional activities on the basis of the evolving local instruction theory. In relation to this we speak of *micro design cycles*, which are very similar to Simon's (1995) HLT: (1) anticipate in advance what the mental activities of the students will be when they will participate in some envisioned instructional activities, (2) try to find out to what extend the actual thinking processes of the students correspond with the hypothesized ones (3) reconsider potential or revised follow-up activities. During the teaching experiment the researchers have to assemble data that allow for the systematic analysis of the learning processes of the students and the means by which that learning was generated and supported.
3. *Retrospective analysis;* since the instructional sequence and the local instruction theory are revised and adapted during the process, a reconstruction of both the instructional sequence and the local instruction theory that are the product of the teaching experiment is needed. Further the teaching experiment may be framed as a paradigm case of more encompassing phenomena, such as: the proactive role of the teacher, the classroom culture, the role of symbols & tools. Here we may use Glaser and Strauss's (1967) the method of constant comparison. By first establishing what happened in a three step procedure; identifying patterns emerging from the data, describing them as conjectures, and looking for confirmations and refutations—in whole dataset. Secondly, by establishing, why this happened; in a similar procedure aiming at finding explanations/causal mechanisms. By first describing them as conjectures, then looking for confirmations and refutations.

Mark that the data analysis needs an interpretative framework to translate observed phenomena in empirical data. In relation to his we may refer to Yackel & Cobb's (1996) emergent perspective. From a methodological perspective, we may further point to the methodological norm of *trackability*, which we take from ethnography: Outsiders should be able to retrace the learning process of the researcher(s). Here we follow Smaling (1992) who points out that the classical methodological norm of *reliability* actually refers to replicability—which in qualitative research translates into virtual replicability. This fits with the goal of offering teacher an empirically grounded theory, which they may adapt it to their own situation by designing HLT's is tailored to their students, and their goals.

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Discourse and Transparency: Avoiding Agnosticism in Our Work with Teachers

Beth Herbel-Eisenmann
Michigan State University

In this paper, I draw on instances of work that I have done with my teacher-researcher and teacher-educator-researcher colleagues in the U.S. and Canada to ask readers to explore and reflect on their own practice as teacher educators. These explorations prompt readers to make transparent some of the underlying Discourses (i.e., associated assumptions, meaning, values, beliefs, and so on) and influencing factors that inform and impact their work. Alongside these instances and explorations, I share some of the important lessons these colleagues have taught me about equity in professional development contexts.

In my title, I write about "avoiding agnosticism." I used to associate the word agnosticism with religion until I read an Editorial by Peter Sullivan in the *Journal of Mathematics Teacher Education*. He pointed out that agnosticism's broader definition is "being uncertain or uncommitted to a certain thing" (Sullivan, 2006, p. 307). I chose this word not because of its focus on uncertainty, which I do not think needs to be avoided because it may be a necessity to reflective practice. Rather, being uncommitted was what I want to advocate we avoid, as mathematics teacher educators. I do not mean to say that I think people in mathematics teacher education are uncommitted, *per se*, because I do think we are committed to quality teacher education, more broadly. Instead, the version of 'uncommittedness' I focus on here is more tacit than that and, thus, something we need to make more transparent. In the same ways that we often ask prospective and practicing teachers to explore their beliefs, values, commitments, and constraints, I think we, too, as teacher educators need to do more of this. My hope is that this paper will prompt readers to explore questions like, *What do you think is at the core of your practices as a mathematics teacher educator? What do you value? At a deeper level, what hidden assumptions and "Discourses" (Gee, 2014) are embedded in these practices, beliefs, and values?* This last question is important because, as Phelan (2015) points out:

> *Discourses organize meanings and practices and allow certain ways of thinking and acting to be considered correct or acceptable, while others are viewed as incorrect or unimaginable (p. 97).*

Moreover, because Discourses underlie what we often treat as "normal" or "common sense," I hope this paper pushes us to make more transparent some of the Discourses that underlie our work. It is only through transparency that we can decide whether what is being treated as “correct” or “acceptable” is, in fact, equitable.

An Introduction Via Analogy

At this point in time, I strongly believe that, like teaching children, teacher education is primarily about relationships. I play out an extended analogy about “teacher education as relationships” in order to set the stage for the remainder of this paper. I do not think of teacher education as being just *any* kind of relationship. Rather I am committed to relationships that are not possible in typical professional development (PD) that takes place in schools all over the U.S. By “typical” PD, I mean the kind where: someone other than teachers (e.g., a curriculum specialist, mathematics specialist, or administrator) decides who to bring in as an ‘expert’ based often on related policies that have been adopted at the state or district level. Typically the PD is made up of a short-term workshop or presentations that focus on telling or showing teachers what they should do. The values and purposes are determined ahead of time, often with little contextual knowledge of the place in which the PD is happening. All of the teachers are expected to attend and sometimes the PD days are built into the school calendar as a way to mandate teacher attendance. Teachers have little agency in this process. This description matches the PD I attended as a teacher, the PD that all of the teachers I have worked with have experienced, and has been described in Judith Warren Little’s (1990) large-scale work in the U.S. as being some of the most common experiences teachers have.

Making use of my extended analogy of teacher education as relationships, my current view on these one-shot workshops can be captured by how I think about brief love affairs. Like brief love affairs, one-shot workshops begin with an initial recognition of someone—an attraction to someone personally (across the room at a party, for example) or professionally (when one receives a phone call from a school where they explain why you are the perfect person to come to do the PD) for their expertise. The potential for pleasure may feel worth it: the excitement of the pursuit and novelty in the former case; the offer of a nice stipend for the two hours work with teachers and, if you are a dynamic speaker, the overhearing of teacher comments who excitedly leave the room, talking about how interesting the PD was. Quickly, however, the reality can set in: the lack of commitment, the potential dishonesty. The “I’ll call you tomorrow” that never happens; the, “Oh, yea, that was interesting, but I don’t have time for that” talk from teachers in the hallways. The bigger Discourses that might get perpetuated about women, in the one case, and about academics and teachers and their work, in the other. I stop here for brevity’s sake. I want to emphasize that there are other kinds of relationships and other kinds of short- and long-term consequences to work with teachers. In the following sections, I return to this broader analogy of teacher education as relationships to frame some of the examples I share from my work.

Two Inspirations for My Work

This stance was not my original stance about teacher PD, rather it is a stance I have developed as I have collaborated with teachers, in honour of their perspectives, the relationships I have developed with them, and the “ah-ha!” moments I have experienced working with them. This stance also has developed in resistance to the fact that the typical PD experience described above is still something teachers experience every year of their professional lives. The teachers with whom I have collaborated have found these experiences quite disempowering. As one of them said:

> *We’re just never, ever, ever, ever, ever treated with autonomy or to think that what we think would be best,*
> *Or to think about what’s important and do it for a long time,*
> *Or to be supported in what you think is best over a long time…that structure [of being part of a study group and doing action research over three years] was so foreign to me. (Teacher-Researcher Interview)*

This kind of relationship—not being treated with autonomy, being treated as if you do not know what is best, and feeling that you are not supported—is not a healthy one. In fact, if we examine the kind of Discourses embedded in this pervasive practice, we see: someone (other than teachers) knows what teachers need to know and do; all the teachers need the same thing; context is not important; teaching is fairly simple because it can be broken down into things one *does*, teachers only need to follow someone else’s suggestions to teach better; and the process of enacting those suggestions is simple so no follow-up is needed. Typical ‘professional’ development experiences, I would argue, perpetuate Discourses that de-professionalize teachers and teaching.

The second inspiration for the work that I do might be captured in terms of a relationship like an “overly critical parent” or the person who is outside of some experience you have and, when you talk about issues you may have related to this experience, this person mainly critiques and points out insufficiencies, but offers little or no suggestions for what to do differently. This is a bit of an overstatement, but when I was first introduced to research on mathematics classroom discourse, I was surprised at the overly critical tone in some of this work. I had just recently been teaching grades 7-9 mathematics and, given the critical nature of this work, looked for more information about what teachers might do differently. As a secondary mathematics teacher, I had a lot of coursework in mathematics but had never been exposed to information like that which I read in this literature. I wondered why we would think mathematics teachers would do anything different from what was being reported.

I also noticed that there was little focus in this research on the role of 'common sense' in the discourse practices and that there were no descriptions of collaborations with the teachers to work through dilemmas and issues with them. Instead, the articles were more distance reports of what happened in the classroom discourse. Some of the Discourses that these kinds of reports could perpetuate include: delegitimising 'insider' perspectives (Cochran-Smith & Lytle, 1993), maintaining a divide between research and practice, and assuming that teachers should 'step out' of and question their own Discourses in ways that most people, in general, do not. Some of the work that I have done over the past 10 years has been in response to this "overly critical parent" relationship.

Background on Collaboration

As way of background, I describe a long-term collaboration I had with a group of eight secondary mathematics teachers and Michelle Cirillo, who was a PhD student at the time. From 2004-2010, we worked together to better understand how mathematics teachers' beliefs and practices might change over time when they were part of a study group reading about mathematics classroom discourse and engaging in cycles of action research (see Herbel-Eisenmann and Cirillo, 2009). We spent a year (2005-2006) collecting base-line data on each teacher's background, beliefs, and current mathematics classroom discourse practices through video-recording four weeks of their classroom interactions across the school year. We then read and discussed many books, articles, and book chapters focused on classroom discourse and mathematics classroom discourse. After a few months of reading about action research, each teacher designated a focus for her/his first cycle of action research and then spent two years engaging in cycles of action research (including collecting and analysing data, reading additional literature, and reframing the focus as appropriate). The teacher-researchers also provided member checks for analyses that we did related to the overarching project goals.

In the next few sections, I share two investigations of the study group setting and related contemplations about my practice as a mathematics teacher educator. The investigations relate to my trying to develop different kinds of relationships than typical PD in my work with teachers. The contemplations will make transparent some of the things I have learned from this work. I believe that this kind of contemplation of our practices as mathematics teacher educators can lead us down a path of Discourses that forge relationships and advocate for the professionalism of teaching, which could lead to a more equitable treatment of teachers, more broadly. I return to this point later.

Investigation 1: Professed Beliefs and Practices

In this first investigation, I focus on a relationship that only became apparent to me after study group data were analysed. The relationship I think this captures is that of an "unreflective mentor" or working with someone who is maybe more experienced than you are, but who mentors in the same way s/he was mentored. It was not until a few years into the project that I made explicit to myself some of my own professed beliefs about work with teachers going into the project. This reflection was prompted by the fact that, in the fourth year of the project, I began to wonder whether I was being as helpful as I might be to the teachers as they investigated and tried to be more purposeful about their discourse practices. One belief that I went into this work with was that teachers' practical knowledge was different from, but just as important as, knowledge published in academic journals. For example, teachers' practical knowledge is often more contextually grounded, localized, nuanced, and meaningful to practice than knowledge published in academic journals. Its standards are guided by trustworthiness rather than some form of 'validity' as is often described in academic research (Zeichner & Noffke, 2001). Like others who engage in collaborative teacher research, I want to challenge "the hegemony of an exclusively university-generated knowledge base for teaching" (Cochran-Smith & Lytle, 1999; c.f. Atweh, 2004; Cochran-Smith & Lytle, 1993; Zeichner & Noffke, 2001).

Lord (1994) had also convinced me that engaging in "critical colleagueship" was necessary for transforming practice. Lord's framework that described critical colleagueship included, for example:

- *Creating and sustaining productive disequilibrium through self-reflection, collegial dialogue, and on-going critique.*
- *Embracing fundamental intellectual virtues (such as openness to new ideas, willingness to reject weak practices or flimsy reasoning when faced with countervailing evidence and sound arguments, accepting responsibility for acquiring and using relevant information in technical arguments, assuming collective responsibility for creating a professional record of teachers' research and experimentation) (pp. 192-193).*

In particular, these notions about critique and how one expresses "intellectual virtues" were compelling to me and I tried to work on them in the study group and during discussions of my collaborators' action research projects.

I was fortunate to have these aspects of my practice as a teacher educator interrogated when two PhD students with whom I worked agreed to investigate the PD interactions (see Males, Otten, & Herbel-Eisenmann, 2010). The research question addressed was: What are the features of challenging interactions in each of the phases of the PD (i.e., study group discussions versus action research discussions) and how do challenging interactions relate to critical colleagueship, in particular, intellectual virtues, found? "Challenging interactions" were defined as interactions in which the teacher-researchers or teacher-educator-researchers probed or used questions to push individuals to think more deeply about an idea or a particular practice. Very briefly, we found: 1) There were many more challenges during study group discussions than during action research discussions; and 2) The teacher-researchers used stories as the basis for their reasoning, rather than a form of argumentation.

What did this investigation make me contemplate as a mathematics teacher educator? This idea of "intellectual virtues" involves practices more like academic university discourse than discourse practices outside of academia. I realized that my enculturation into academic culture made me value these virtues in new ways; they were not something I experienced in PD as a classroom teacher. I have now come to think of these project meetings as a "hybrid space"—not a course in which I am trying to mentor graduate students to become researchers, but also not an informal dinner party with friends.

Given this new view of project meetings as hybrid spaces, I began to question my assumptions about what intellectual virtues might look and sound like. My unarticulated belief going into the project was that teachers would engage in the kind of practices that I learned in academic settings. If I chose to maintain this belief, however, then how was I to make sense of the fact that the teachers chose to use story telling as a form of reasoning? If I continued to maintain this view of intellectual virtues, then what might be some of the unintended consequences of this belief? I turned to the discourse literature to better understand what difference this might belief might make. Two potential unintended consequences emerged from my reading.

First, in literature about "floor" development, I learned that there are at least two kinds of floors: singly-developed floors (SDFs) and collaboratively-developed floors (CDFs) (Edelsky, 1993). Singly-developed floors are characterized by "monologues, single party control, hierarchical interaction where turn takers stand out from non-turn takers and floors are won or lost (Edelsky, 1993, p. 221), whereas collaboratively-developed floors include "more informal, cooperative ventures which [provide] both a cover of 'anonymity' for assertive language use and a comfortable backdrop against which [participants] can display a fuller range of language" (Edelsky, 1993, p. 221). The evidence-based argumentation discourse I had expected had much in common with SDFs. More importantly, I learned that research has shown that men, in mixed-gender meetings, participate more equally with women during CDFs rather than dominating the floor, as they were found to do in SDFs. During CDFs, women were also shown to take on the role of questioner in ways that they did not in SDFs. During CDFs, then, women and men might be more likely to interact as equals than they might in meetings that are based on SDFs. It may be that SDFs that are characteristic of academic discourse are not appropriate for these hybrid spaces and they may perpetuate gender inequity.

One reaction to this finding might be that I needed to change my practice by establishing different norms for interaction that are more like academic argumentation. For example, I could make the norms of intellectual virtue more explicit, explaining to teacher-researchers how to interact in ways that might be valued by academic researchers. Yet, as someone who values practical knowledge, this option seems limited and might de-value practical knowledge. From the discourse literature, I also learned that narratives play an important role in developing complex understandings and are important persuasive tools. For example, Florio-Ruane (2001) examined the narratives that prospective teachers told during a study group focused on culture and literacy and found that narratives were part of an important, intellectual process that helped prospective teachers learn about themselves and their role in teaching. Through discussions, the narratives built upon one another and moved the joint work of the group forward, acting as a scaffold with peers and/or more experienced others and resulting in a deeper understanding of culture and identity. Florio-Ruane argued that participants formed “a kind of connected knowing” (p. 136) through their narratives. Furthermore, I learned from Juzwik (2009) that narratives serve a performance function by which teachers were able to identify with others in order to persuade them. Attending to stories as rhetorical devices could help me to understand how narratives persuade in more subtle ways than explicit claims-evidence argumentation.

I now understand that I need to pay closer attention to how the floor is being developed in project meetings and that I need to listen carefully to when, how and why teacher-researchers tell the kinds of stories they do. I also need to consider the ways in which these stories construct a complex understanding of classroom discourse through tracing how the narratives build on the thematics of each other. If stories are important sense-making tools for teachers, then my practice should develop toward knowing when stories stall the work or when they help us move forward. According to Florio-Ruane (personal communication, January, 2010), these skills can be developed through careful listening, discussions with teachers, continued reflection, and systematic investigation.

Investigation 2: Exploring Discourse-Related Ideas *with* Teachers

This second investigation (see Herbel-Eisenmann, Drake, & Cirillo, 2009) of the study group interactions is an example of my trying to work in opposition to the “overly critical parent” relationship I described earlier. This example illustrates what might happen when we develop a collaborative relationship with teacher-researchers over a period of time and how those relationship can help us see anew particular ideas that are primarily written about in articles by university researchers.

One central discourse-related idea that the teacher-researchers in this project talked about and became interested in exploring in their practice was the idea of "revoicing" or "the reuttering of another person's speech through repetition, expansion, rephrasing, and reporting" (Forman, Larreamendy-Joerns, et al., 1998, p. 531; originally introduced by O'Connor & Michaels, 1993; 1996). Mathematics education researchers have labelled this idea as "powerful" (e.g., Franke, Kazemi, & Battey, 2007) -especially in relation to the more typical Initiate-Respond-Evaluate (Mehan, 1979) interaction pattern. The reported evidence related to the impact of revoicing, however, is sparse. We also know little about how teachers and students think about various discourse moves, including revoicing. I would argue that it is difficult to know how revoicing might be powerful unless we understand how it is interpreted by teachers and students, the people who are actually engaged in these discourse practices.

In this investigation, we took seriously the need for mathematics education researchers and teacher educators to better understand revoicing from teachers' perspectives. The questions we sought to answer were: What did the teacher-researchers talk about when they talked about revoicing? How did they talk about revoicing? How did their ways of talking about revoicing change over time?

We found that the teacher researchers highlighted the multiple (and often simultaneous) forms, functions and meanings of revoicing. They also recognized the fact that revoicing could have intended and unintended meanings. From their insider perspective, they worried that students may have different interpretations than they did, as teachers and adults in the classroom. Many of the functions they identified related to issues of authority, power, control, and ownership of ideas. For example, teachers repeatedly distinguished situations when repeating a student might be appropriate versus when rephrasing might be appropriate. The distinction between repeating and rephrasing was related to some of the dilemmas they faced in thinking about revoicing in their own classrooms. The teacher-researchers seemed to associate repeating with allowing students to maintain ownership of their ideas, whereas rephrasing seemed to shift the ownership or control from the student to the teacher. They especially worried that, if they rephrased too much, the students would no longer see the ideas as their own. On the other hand, rephrasing was also seen as an initial step toward helping students gain facility with mathematical discourse, whereas repeating seemed more often to serve the purposes of amplification or encouraging students to listen to each other's ideas. Finally, the teacher-researchers highlighted the fact that the context mattered to their interpretations of revoicing. Their interpretations, for example, focused on contextual information like which class period they were in, the nature of prior experiences they had with particular students, and where they were in relationship to developing content ideas.

In contrast to these issues the teacher-researchers raised, publications written by university researchers tended to do quite different things. Typically, for example, university research clarifies and defines complex phenomenon in ways that reduce the messiness of the ideas. The interpretations they offer are from their outsider perspective view, often drawing on particular theoretical or discursive frameworks that make sense to them. Finally, the only place "context" appears in these publications is in the methods section when researchers describe the research context. The kinds of context the teacher-researchers described as being important is typically not included in the findings of research articles.

This investigation helped me to contemplate the fact that practical knowledge unearths the messiness and nuance of teachers' work. Knowing more about teachers' perspectives help me to better understand what they do and why they do it. I also learned that the lenses being brought to the work on revoicing (and many other discourse-related constructs) needed to be augmented to account for issues of power, control, ownership and authority. These shifts in my thinking had a strong impact on my work since 2010, for example, inspiring another long-term collaboration with David Wagner and a group of teachers in Canada to understand issues of authority (Wagner & Herbel-Eisenmann, 2014a; 2014b). From a more practical stance, this inquiry also provided a useful process for reflecting on and improving my practice as a teacher educator and an author of PD materials (Herbel-Eisenmann, Cirillo, Steele, Otten & Johnson, forthcoming). We have since then used this process to improve a set of professional development materials we have been developing and piloting (c.f. Herbel-Eisenmann, Steele, & Cirillo, 2013). After some of the piloting of these materials, we investigated how teachers talked about the mathematics register (Herbel-Eisenmann, Johnson, Otten, Cirillo, & Steele, 2015) and about positioning in their classrooms (Cavanna & Herbel-Eisenmann, 2014; Suh, Theakston Musselman, Herbel-Eisenmann, & Steele, 2013). We were able to use what we learned to improve the activities and the support materials for facilitators.

Conclusion: Obligation to Relationships and Advocacy

The two investigations and contemplations I shared have brought me to a clearer understanding of the kinds of relationships I am trying to work toward and how to work toward what I now think of as "healthy relationships" with teachers. These relationships are based on genuineness, honesty, commitment from both sides, support, communication, and not on false pretense or promises to 'fix' someone or something. They require on-going reflection on what aspects work in the relationship, for whom, and why.

I see an obligation now to work toward healthy relationships because they are a form of advocacy for teachers. Although we often talk to teachers about advocating for the children with whom they work, as teacher educators, we have to consider our own power to advocate for teachers, too. In a policy context like the U.S. in 2001 where national leaders put into place No Child Left Behind (a program that substantially increased the testing requirements to increase accountability (see Linn, Baker, & Betebenner, 2002)), state policymakers proclaim that teachers are just glorified babysitters and need to be monitored through things like students' test scores, and neoliberalism is the underpinning view guiding decision-making, I see an obligation to take a stance of commitment to stand up for teachers. I believe we need to: let policymakers and decision-makers know and understand that one-shot workshops are not healthy relationships nor are they ways to support teachers to improve their practice; say no when districts call us to do one-shot workshops and explain to them why we are saying no; and to resist these Discourses of de-professionalisation, despite the potential "feel good" things I mentioned when I compared them to brief love affairs.

I end with a declaration and plea for mathematics teacher educators to *avoid agnosticism*. Some of the ways we can get started are to: make explicit the analogies/metaphors of our work in order to make transparent our (often tacit) beliefs and values; unearth the embedded assumptions in order to understand the Discourses we are treating as common sense so we can make purposeful decisions about disrupting inequitable practices; explore phenomenon *alongside* teachers; and commit to healthy relationships and advocacy. If we expect teachers to work toward equitable practices with their students, we must also work toward more equitable systems and practices for teachers.

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Foundational Number Sense: A Framework For Analysing Early Number-Related Teaching

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In this paper, by means of an extensive review of the literature, we discuss the development of a framework for analysing the opportunities, both implicit and explicit, that grade one students receive for acquiring those number-related understandings necessary for later mathematical achievement but which do not occur without formal instruction. The framework, which we have called foundational number sense, currently comprises seven interrelated components, although additional components may exist. Each component, as warranted by earlier research, is known to underpin later mathematical understanding and, when viewed collectively, addresses a definitional gap in the literature.

Introduction

In an earlier paper (Back, et al., 2013) we introduced and evaluated the efficacy of a framework for identifying the learning opportunities, both implicit and explicit, pupils receive for acquiring foundational number sense. Derived from the literature, this tentatively proposed framework was not only able to identify opportunities linked to those basic number competences thought to be necessary for successful mathematical learning but was sensitive to culturally different teaching traditions. In this theoretical paper we offer an extended account of the derivation of this framework.

Described as a "traditional emphasis in early childhood classrooms" (Casey et al. 2004: 169), children's acquisition of number sense is acknowledged as a key objective of many early years' mathematics curricula (Howell & Kemp 2005; Yang & Li, 2008). It is not only a predictor of later mathematical success, both in the short (Aubrey & Godfrey, 2003; Aunio & Niemivirta, 2010) and the longer term (Aubrey et al., 2006; Aunola et al., 2004), but brings numbers to life and enhances our relationships with them (Robinson et al., 2002).

While number sense "is considered internationally to be an important ingredient in mathematics teaching and learning" (Yang & Li 2008, p.443), there is evidence that it is gender-determined, with boys typically outperforming girls on standard measures at ages five and six, a difference compounded by parental education levels – the more highly educated the parents the better boys perform (Melhuish et al., 2008; Penner & Paret, 2008). On the other hand, evidence shows that number sense is gender-independent, although there are cultural differences, with, for example, Chinese children exhibiting higher levels of counting skills than Finnish students, irrespective of age (Aunio et al., 2006). Of course, such research inconsistency may be a consequence of differences in the measures used. Where research seems to be consistent is in the influence of various components of the socio-economic status of a child's family (Melhuish et al., 2008; Starkey et al., 2004). Indeed, without appropriate intervention, which research shows can be effective (Van Luit & Schopman, 2000), children who start school with limited number sense are likely to remain low achievers throughout their schooling (Aubrey et al., 2006).

What is number sense?

While it is important to understand the consequences of poorly or inappropriately developed number sense, it is equally important that we have a clear understanding of what is meant by the term. In this respect, the National Council for Teachers of Mathematics has written, somewhat vaguely, that it is "an intuition about numbers that is drawn from all varied meanings of number" (NCTM, 1989, p.39). Others have been equally imprecise, as with, for example, the definitions of Case (1998), Griffin (2004) and McIntosh, Reys and Reys (1992). Indeed, despite its apparent importance, "no two researchers have defined number sense in precisely the same fashion" (Gersten et al., 2005, p.296), which would clearly make the development of classroom interventions problematic.

Interestingly, Berch (2005) has argued that such ambiguities are compounded by the fact that psychologists and mathematics educators work to different definitions, a dichotomisation exacerbated by our interpretation of the former literature, whereby researchers differ according to whether they work in the fields of general cognition or learning disabilities. Irrespective of such research traditions, our reading of the literature reveals two distinct perspectives on number sense. The first, which we have labelled foundational number sense, concerns the number-related understandings children develop during the first years of formal instruction. The second, which we have labelled applied number sense and which incorporates the first, concerns the number-related understanding necessary for people to function effectively in society. Students with a well-developed applied number sense

> will look at a problem holistically before confronting details, look for relationships among numbers and operations and will consider the context in which a question is posed; choose or invent a method that takes advantage of his or her own understanding of the relationships between numbers or between numbers and operations and will seek the most efficient representation for the given task; use benchmarks to judge number magnitude; and recognize unreasonable results for calculations in the normal process of reflecting on answers (Reys, 1994, p. 115).

Such behaviours underpin what is known as adaptive expertise (Hatano & Inagaki, 1986). Adaptive experts have the flexible understanding, structured by the principles of the discipline (Pandy et al., 2004), necessary for solving non-routine problems. They not only modify or invent procedures (Hatano & Inagaki, 1986) but self-regulate their learning as a dynamic rather than static entity (Martin et al., 2005; Verschaffel et al., 2009). Adaptive expertise requires an appropriately deep conceptual knowledge to give meaning to the procedures taught (Hatano, 1982). In this paper, while mindful of the form and function of applied number sense, we focus on foundational number sense as the basis for much later teaching.

Defining foundational number sense

Foundational number sense is to the development of mathematical competence what phonic awareness is to reading (Gersten & Chard, 1999), in that early deficits tend to lead to later difficulties (Jordan et al., 2007; Mazzocco & Thompson, 2005). Significantly, it has been shown to be a more robust predictor of later mathematical success than almost any other factor (Aunio & Niemivirta, 2010; Byrnes & Waski, 2009).

So, what are the characteristics of foundational number sense? Broadly speaking it is the ability to operate flexibly with number and quantity (Aunio et al., 2006; Clarke & Shinn, 2004; Gersten & Chard 1999) and can be expressed as attributes like "awareness, intuition, recognition, knowledge, skill, ability, desire, feel, expectation, process, conceptual structure, or mental number line" (Berch 2005, p. 333). In particular, there is evidence that elements of number sense are innate to all humans and independent of instruction. This *preverbal* (Ivrendi, 2011; Lipton & Spelke, 2005) component comprises an understanding of small numbers in ways that allow for comparison. For example, Feigenson et al. (2004, p. 307) found that "6-month-olds can discriminate numerosities with a 1:2 but not a 2:3 ratio, whereas 10-month-old infants also succeed with the latter", adding that "adults can discriminate ratios as small as 7:8". Thus, as Lipton and Spelke (2005, p.978) observe, "numerical discrimination becomes more precise during infancy... but remains less precise than that of adults". This preverbal number sense is independent of formal instruction, developing in the early years as an innate consequence of human, and other species', evolution (Dehaene, 2001; Feigenson et al., 2004).

Later, but still before the start of formal school, and frequently dependent on individual family circumstances, children as young as three can undertake, without instruction, addition- and subtraction-related tasks with confidence and accuracy (Zur & Gelman, 2004). By age four children have normally acquired initial counting skills and an awareness of quantity that allows them to respond to questions about 'more' and 'less' (Aunio et al., 2006). At about the time they start school children typically acquire a sense of a mental number line, including “knowledge of number words, the ability to point to objects when counting, and knowledge of cardinal set values” Aunio et al., 2006, p.484). However, although there are indicators of a typical developmental trajectory, the properties of foundational number sense remains vague. In this paper our interest lies not with the preverbal number sense described above but the number sense that requires instruction (Ivrendi, 2011). Our reading of the literature leads us to conclude that there are seven, although there may be more, interrelated components, which are:

1. Foundational number sense involves number recognition, its vocabulary and meaning (Malofeeva et al., 2004). It entails being able to both identify a particular number symbol from a collection of number symbols and name a number when shown that symbol, typically up to twenty (Clarke and Shinn, 2004; Van de Rijt et al., 1999; Gurganus, 2004; Yang & Li, 2008). Significantly, children who experience difficulty with number recognition tend to experience later mathematical problems generally (Lemke & Foegen, 2009) and particularly with subitising, a key process of early arithmetic (Koontz & Berch, 1996; Stock et al. 2010).

2. Foundational number sense incorporates systematic counting (Berch, 2005; Clarke & Shinn, 2004; Van de Rijt et al., 1999; Griffin, 2004). It includes notions of ordinality and cardinality (Ivrendi, 2011; Jordan et al., 2006; Jordan & Levine 2009; LeFevre et al., 2006; Malofeeva et al., 2004) and, in particular, the learning of the order of the various number names (Van Luit & Schopman, 2000). Typically, children can count to twenty and back or count upwards and backwards from an arbitrary starting point (Lipton & Spelke, 2005), knowing that each number occupies a fixed position in the sequence of all numbers (Griffin et al., 2004). Significantly, unsophisticated counters may have later difficulties developing adaptive solution strategies for the various arithmetical problems they encounter (Gersten et al., 2005; Stock et al., 2010).

3. Foundational number sense includes an awareness of the relationship between number and quantity. In particular, children understand not only the one-to-one correspondence between a number’s name and the quantity it represents but also that the last number in a count represents the total number of objects (Malofeeva et al., 2004; Van Luit & Schopman, 2000). It incorporates quantity discrimination, whereby children recognise that eight represents a quantity that is bigger than six but smaller than ten (Gurganus, 2004; Lembke & Foegen, 2009). Importantly, quantity discrimination is a predictor of a child’s later mathematics achievement (Kroesbergen et al., 2009).

4. At the foundational level, number sense includes awareness of magnitude and of comparisons between different magnitudes (Case, 1998; Clarke & Shinn, 2004; Griffin, 2004; Gurganus, 2004; Ivrendi, 2011; Jordan et al., 2006; Jordan & Levine 2009; Yang & Li, 2008) and deploys language like ‘bigger than’ or

'smaller than' (Gersten et al., 2005). In particular, children who are magnitude aware have moved beyond counting as "a memorized list and a mechanical routine, without attaching any sense of numerical magnitudes to the words" (Lipton & Spelke, 2005, p. 979). Moreover, being magnitude aware supports the development of other mathematical skills, particularly subitising (Aunio & Niemivirta, 2010; Nan et al., 2006; Stock et al., 2010).

5. A foundational number sense aware child is able to estimate, whether it be the size of a set (Berch, 2005; Gersten et al., 2005; Jordan et al., 2006, 2007: Malofeeva et al., 2004; Van de Rijt et al., 1999;) or an object (Ivrendi, 2011). Estimation involves moving between representations - sometimes the same, sometimes different - of number, for example, placing a number on an empty number (Booth and Siegler, 2006). However, the skills of estimation are dependent on the skills of a child to count (Lipton and Spelke, 2005).

6. A foundational number sense aware child will be able to perform simple arithmetical operations (Ivrendi, 2011; Jordan & Levine 2009; Yang & Li, 2008); skills which underpin later arithmetical and mathematical fluency (Berch, 2005; Dehaene, 2001; Jordan et al., 2007). Indeed, simple arithmetical competence, which Jordan and Levine (2009) describe as the transformation of small sets through addition and subtraction, has been found to be, at grade one, a stronger predictor of later mathematical success than measures of general intelligence (Geary et al., 2009).

7. Foundational number sense includes awareness of number patterns and, in particular, being able to identify a missing number (Berch, 2005; Case, 1998; Clarke & Shinn, 2004; Gersten et al., 2005; Gray & Tall, 1994; Jordan et al., 2006, 2007). Such skills reinforce the skills of counting and facilitate later arithmetical operations (Van Luit & Schopman 2000). Importantly, failure to identify a missing number in a sequence is a strong indicator of later mathematical difficulties (Chard et al., 2005; Clarke & Shinn, 2004; Gersten et al., 2005; Lemke & Foegen, 2009).

The development of foundational number sense

How a child comes to acquire number sense is complex and, in some ways, circular. For example, Malofeeva et al. (2004) argue that counting and knowledge of numerical symbols underpin the development of number sense concepts, and yet these are themselves components of number sense. That being said, "there is general agreement that number sense is a construct that children acquire or attain, rather than simply possess" (Robinson et al., 2002, p. 85); therefore, it would seem sensible to assume it does not occur by chance but "requires a conscious, coordinated effort to build connections and meaning on the part of the teacher" (Reys, 1994, p. 115). In general, this means that teachers should, inter alia, encourage children to work with concrete materials and familiar ideas; discuss and share solutions and discoveries; compose and recompose different representations of numbers; explore number patterns and number relationships; create alternative methods of calculation and estimation (Griffin, 2004; Tsao & Lin, 2012). Such invocations resonate well with the characteristics described above. Moreover, in the light of evidence that young children from high-socioeconomic status (SES) backgrounds are five times as successful as children from low SES backgrounds on tasks like, 'which number is bigger, 5 or 4?' (Griffin, Case, & Siegler, 1994), the case for intervention seems clear, particularly as "aspects of number sense development may be linked to the amount of informal instruction that students receive at home on number concepts" (Gersten et al., 2005, p. 297). Importantly, "number sense develops gradually over time as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms" (Sood & Jitendra, 2007, p. 146).

Issues in foundational number sense

The consequences of poor number sense are significant. For example, basic counting and enumerations skills have been found to be predictive of later arithmetical competence in England, Finland, Flanders, USA, Canada and Taiwan respectively (Aubrey & Godfrey, 2003; Aunola et al., 2004; Desoete et al., 2009; Jordan et al., 2007; LeFevre et al., 2006; Yang & Li, 2008). In other words, there is an international consensus that poorly developed number sense underlies later mathematical failures (Jordan et al., 2009; Gersten et al., 2005; Malofeeva et al., 2004). There is also evidence that teachers may have contributed to their children's difficulties. For example, while children's counting competence increases with age, their tolerance of unusual counts, that is counting procedures that do not progress naturally from left to right, diminish, leading to the conclusion that the ways in which they are typically taught may be counter-productive in terms of establishing an awareness that the order of a count is an inessential element of the process (LeFevre et al., 2006). Moreover, as Wagner & Davis (2010, p. 40) note, an emphasis on an understanding of quantity in the early years of schooling is so eclipsed by later expectations of computational competence that "students become numb to the meaning of the numeric symbols they learn to manipulate".

Conclusions

In this essentially theoretical paper we have explicated a framework for analysing classroom activity in the early years of mathematics teaching. This is a novel undertaking as earlier evaluative studies have focused on children's competence and not the opportunities teachers provide for them. Importantly, the framework has been piloted on two lessons, one from Hungary and one from

England (Back et al, 2013), and found effective in identifying the number sense opportunities presented by the teachers concerned. However, the reader is reminded that the validity of each of these seven components can be found in the literature from which it derives. That is, the importance of each component in the subsequent development of children's mathematical competence has been warranted by the research that informed its inclusion here. What has yet to be done, beyond the initial trial discussed above, is an assessment of the framework's efficacy as a tool for analysing the opportunities teachers offer their students, to be followed by an analysis of different ways in which they do this.

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How Research Conceptualises the Student in Need of Special Education in Mathematics

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The focus of this paper is the conceptualisation of students in special educational needs in mathematics (SEM students) in the research fields of mathematics and special education. A difference between fields regarding the perspectives taken on the SEM student is obvious in the reviewed articles. Those in the special educational field were individual oriented in their view of the difficulties, whilst reviewed articles from the field of mathematics education more often discuss socio-cultural settings. The content in the selected 28 articles reveals that the overall conceptualisation of SEM student has to do with the social construct of the SEM student, as well as with students' experiences, affects, and prerequisites; with the specific training methods or interventions applied; with special areas in the subject of mathematics; with special groups of students; and with teachers' knowledge about all these factors.

Introduction

Research is grappling with the concept of students who are in need of special education in mathematics (SEM-students; Magne, 2006). Despite a sustained debate in various fields on how to help the student in need, there is no shared understanding (Heyd-Metzuyanim, 2013). Challenges involving the conceptualisation of the SEM student are similar to the differences McLeod and Adams (1989) describe in the use of the concept *affect* by mathematics educators and psychologists. Is it then possible that researchers that use the same concept mean different things, or researchers use different concepts but mean the same thing? Clarifying the concept could decrease the risk of misinterpretation and misconceptions. The present study contributes to an investigation regarding ways that the SEM-student is conceptualised in research. We build on a pedagogical foundation in the understanding of the SEM-student, since it is in the mathematics educational setting that the need occurs which is later handled by special pedagogical approaches. An effort to emphasise the student in the educational context makes the fields of mathematics education and special education sufficient research areas to explore. We analysed how research defines the SEM-student by identifying parts in the articles which *conceptualise* the student in need and explain the *cause of difficulties* and what *kind of support* is given in order to facilitate learning. The article at hand conceptualise the SEM-student as being about the individual's *need* for special education. We draw on Silfver, Sjöberg and Bagger (2013) in the understanding of the position of need as something that may occur whether the student is a high or a low achiever, for a short or a long period, in a general or more specific area in mathematics.

A conceptual framework for categorisation

Perspectives involved in research on the student in need of support traditionally involve several fields of expertise, which are connected to a psychological, social, or pedagogical field (Emanuelsson, Persson, & Rosenqvist, 2001; Heyd-Metzuyanim, 2013; Isaksson, 2009; Magne, 2006; Nilholm, 2005; Persson, 2008; Skrtic, 1995). In addition, several levels and actors are involved when a school educates a student in need (Ahlberg, 2001, 2007; Skrtic, 1995). Nilholm (2005, 2007b) has labelled perspectives on special education as *compensatory* or *critical*, categories similar to those Persson (2008) calls *categorical* and *relational*. With both the critical and the relational perspectives, the heritage of the problem is described as located in socio-cultural settings. Solutions are then found by adapting the learning environment and the relations that surround the SEM student. A categorical or compensatory perspective in special education places the problem inside the student and can be described as a deviation from what is "normal." Training, compensation, and correction of the individual are then necessary. Nilholm (2005, 2007a, 2007b) has furthermore described a third perspective that allows an evaluation and critique of both the compensatory and the critical perspectives used in research: the *dilemma perspective. Dilemma* (Nilholm, 2005, 2007b) refers to the unsolvable and contradicting problems involved in special pedagogical practice. Dilemmas can appear when the motives for supporting the student contravene the demands of society or the school system. In this paper the categorical and relational perspectives have been used in the categorisation of articles, and the dilemma perspective has been used in discussing the content of selected articles.

Methodology and methods

Magne (2006) presents research concerning the SEM student in a special issue of the Nordic mathematics education journal *NOMAD*. The present paper contributes by further reviewing ways that the SEM student is conceptualised and briefly reviews and discusses selected articles. The study focuses on two selected fields of research—namely, *special education* and *mathematics education*. Articles were selected from journals in the areas of mathematics education, special needs, and special pedagogy for the years 2006 to 2013. Journals were initially found through a journal search for the terms *special education* and *mathematics education*. We thereafter selected journals that were peer reviewed and determined their value by guidance from reported impact factors during 2012 in Scopus (Table 1). To further confirm the value, we also used the impact factor in the database Journal Citations Report (JCR). Values of 0.5 or below in JCR are considered low, and values of 1.5 or above are high. In addition to this, we added one journal, which is of importance for our text: The journal *NOMAD* is not listed in the databases but is ranked as a number one in the Database for statistikk om høgre utdanning (DBH). *NOMAD* is important because it represents the Nordic context of mathematics education. In total, 12 journals in the fields of special education (6) and mathematics education (6) were selected. Terms used for searching articles differed between the two fields since they have different focuses on the SEM student. The search term in the special educational journals was *math*, and in the mathematical journals the search words related to special needs: *dys-*, *need*, *support*, *disabilit-*, and *special*. After deselecting articles that did not mention the SEM student in the title or abstract, 28 remained for review.

Table 1. Articles found and used in journals, 2012. With impact factors.

Journal	JCR[1]	SNIP[2]	Country	Issue/ year	Publisher	Index category in SCOPUS	Found	Used
Mathematics Education Research Journal		0.760	Netherlands	3	Springer	Mathematics Social Sciences: Education	0	0
Educational Studies in Mathematics	0.765	1.874	Netherlands	9	Springer	Mathematics, Social Sciences	8	7
NOMAD			Nordic countries	4	NCM		4	4
Research in Mathematics Education		0.315		3	Routledge	Mathematics Social Sciences: Education	5	1
JRME (Journal for Research in Mathematics Education)	1.552	2.782	United States	5	Nat council teach. math.	Mathematics: Mathematics (miscellaneous) Social Sciences: Education	5	0
ZDM (Zentralblatt für Didaktik der Mathematik)		0.676	Germany	6–7	Springer Verlag	Mathematics, Social Sciences: Education	0	0
European Journal of Special Needs Education		1.104	England	4	Blackwell	Psychology: Developmental and Educational Psychology; Social Sciences: Education	5	1
Journal of Special Education	1.278	1.679	England	4	Sage	Medicine: Rehabilitation Social Sciences: Education	1	1
International Journal of Special Education		0.278	Canada	3	International J of Special Edu.	Medicine: Rehabilitation Social Sciences: Education	5	4
Journal of Research in Special Educational Needs		0.773	England	3	Blackwell	Social Sciences: Education	2	2
Remedial and Special Education	0.890	0.795	United States	6	Sage	Medicine: Public Health, Environmental and Occupational Health. Social Sciences: Education	4	4
British Journal of Special Education		0.792	England	4	Blackwell	Psychology: Developmental and Educational Psychology; Social Sciences: Education	4	4
						Total:	**43**	**28**

Analysis

As previously mentioned, the way research defines the SEM student is found by identifying parts in the reviewed articles which *conceptualise* the student in need and explain the *cause of difficulties* and what *kind of support* is given in order to

[1] Journal Citation Reports (JCR). Impact factor from Thompson. The value is based on cites/number of articles from the two previous years.

[2] Source-normalised impact per paper (SNIP). Impact factor from Scopus. The value is based on the number of citations given in the current and three past years, divided by the total number of publications in the past three years—normalisation is made between fields.

facilitate learning. Expressions about these three parts formed the basis for categorising the perspectives used. This was performed using a theoretical framework (see Table 2), drawing on the definitions of perspectives on special pedagogy developed by Nilholm (2005, 2007b) and Persson (2008). Moreover, we discussed this frame for analysing the conceptualisation with Nilholm (personal communication, October 2013). Since some articles lie near both perspectives, it is necessary to clarify boundaries. The application of these borders can be understood as a crossroad in the work of analysis. Accordingly, articles were initially sorted using the concept of the student in need. If an article discussed socio-cultural settings and affect or relations, it was placed in the relational perspective. Articles that found the student through testing and in which interventions were made with a specific child fell under the categorical perspective. If both perspectives were present in the concepts used in an article, we proceeded by looking into the context and the suggested supports and solutions.

Table 2. Frame for analysis

Perspective	Step 1: Concept used	Main cause of difficulty	Step 2: Support or solution
Relational	Describes the environment, relations between pupil's properties and context	Outside the pupil	Changes in the learning environment and relations between pupil and context
Categorical	Describes the pupil's prerequisites or properties	Within the pupil	Strengthen the pupil or compensate for deficits

Results

The results are presented in two parts. We first consider perspectives used by researchers whilst conceptualising SEM students. These perspectives are categorised through the framework described above. In addition, we discuss the review of selected articles through the dilemma perspective (Nilholm, 2005, 2007).

Perspectives on the SEM student in research

Perspectives used in research when conceptualising the SEM student are displayed in Table 3. Four crucial results appeared: (1) there is a significant difference between the field of mathematics and special education, (2) in the field of special education the categorical perspective was predominant, (3) in the mathematics educational field the emphasis on socio-cultural settings is apparent, and (4) because of the procedure for selection (we used only journals indexed under mathematics education or special education), relatively few articles were found. During 7 years and in 12 journals only 28 articles explicitly mentioned the SEM student. This is especially striking considering that the issue is frequently debated amongst politicians, researchers, and professionals in the educational field.

Table 3. Categorisation of the reviewed articles

Perspective used / *Articles indexed as*	Categorical	Relational
Mathematics Education	●●	●●●●● ●●●●●
Education, Special	●●●●● ●●●●● ●●●●●	●

Discussing SEM students in research

What the reviewed research conceptualises as an SEM student and what is of importance for these students' learning can be summarised in six themes: (1) the social construct of the SEM student; (2) students' experiences, affects, and prerequisites; (3) the training methods or interventions employed; (4) special areas in the subject of mathematics; (5) special groups of students; and (6) teachers' knowledge about all these factors. Obviously, these themes overlap, and research often handles more than one of them. Here the themes will be discussed through the dilemma perspective. Dilemmas that concern *the social construct of the student* are apparent when students' needs and the needs of the educational system or of the school itself collide. This is displayed by Clausen-May (2007), who explored the SEM student in the context of international surveys. The need for valid measurement tools then conflicts with the student's need to gain access to tests and to be included in test taking. Although Clausen-May's conceptualisation is categorical (*children with needs*), the discussion critiques the ethos in the distributors' handling of the tests, which does not align with the ethos of the school. Another dilemma regarding the social construct of the student in research appears in that identifying the position of being in need is necessary in order to obtain support, even as this position risks marginalising and segregating individuals by identifying them as "not normal". Researchers who display these approaches do so by investigating the socio-economic or socio-cultural settings and their consequences for the SEM student (e.g., Heyd-Metzuyanim, 2013; Wei, Lenz, & Blackorby, 2013). Environment and individual are explored as a complex. This is the case when the development of a disabled identity is researched through a *commognitive approach* (Heyd-Metzuyanim, 2013).[3] Identity is then dependent on how the environment brings out affective and cognitive factors within the individual. Research that scrutinises students' experiences, affects, and prerequisites often focuses on cognition, especially in the mathematics education journals in our selection. Furinghetti and Morselli (2009) investigate this through students' beliefs about the self and the subject. Malmivouri (2006) understands affect as a part of self-reflection, whilst Evans,

[3] A theoretical framework developed by Anna Sfard (2009). The term *commognitive* merges *communication* and *cognitive*.

Morgan, and Tsatsaroni (2006) research emotions as a "charge attached to ideas or signifiers" (p. 209) and do not take the cognitive aspect into account, instead showing an interest in how social identity is constructed by discourse.

Diagnosis comes into play in research about the SEM student quite differently. It ranges from investigating the mathematics learning of students with a specific diagnosis (Abdelahmeed, 2007; Ahlberg, 2006) to making connections between students with different diagnoses and math achievement (Wei et al., 2013). Here the themes of *special groups of students* and *special areas within mathematics* come together. Some articles focus strictly on how the method might strengthen the individuals with deficits in general in mathematics (Barrett & Fish, 2011; Bryant, Bryant, Gersten, Scammacca, & Chavez, 2008; Ketterlin-Geller, Chard, & Hank, 2008). These studies are all considered categorical in their conceptualisation of the SEM student and are to be found within the field of special pedagogics. Students are here discussed as belonging to a group whose members are functionally similar. The method used to help or the approach investigated might concern a specific area—such as, for example, addition (Calik & Kargin, 2010), subtraction (Peltenburg, van den Heuvel-Panhuizen, & Robitzsch, 2012), or fluent computation (Burns, Kanive, & DeGrande, 2012). A dilemma in research within the themes *special groups of students* and *special areas within mathematics* is revealed only when the methods or interventions take place in inclusive settings. This is, for example, seen in research when approaches or methods are judged to fit all students in groups that include SEM students (Barrett & Fish, 2011; Bottge, Rueda, Serlin, Hung, & Kwon, 2007; Gifford & Rockliffe, 2012) or when students with diagnoses are learners in inclusive classrooms (Calik & Kargin, 2010). Individuals are then understood as having variations in abilities and belonging to a multitudinous group of learners. This can be described as a dilemma consisting of issues of categorisation and differentiation. One example of how this might play out in the conceptualisation of the SEM student is Gifford and Rockliffe's (2012) categorical terminology about the student "with severe specific mathematics difficulties (p.2)" that nevertheless focuses on relational issues: "it would be advantageous to have a single pedagogical approach … that was effective for children with varied difficulties. It would be even more advantageous if this approach were also effective for mainstream teaching, and could prevent mathematics difficulties" (p. 12). This dilemma is further explored in some articles about SEM students in the context of inclusion. For example, inclusive education is compared to solo lecturing (Tremblay & Laval, 2013), and Lindeskov (2006) stresses the need to understand the learners' experience. School placement of the student might in itself determine whether the student is "special" (Calik & Kargin, 2010; Méndez, Lacasa, & Matusov, 2008). Méndez et al. (2008) have used placement as a way of selecting informants and employ a relative expression for the SEM student—namely, *children who demonstrate disability.* This expression might be perceived as placing the problem within the individual, but the authors define the source of challenges in a way that shifts this focus: "Disability is regarded as being located in particular types of activity systems and learning cultures rather than within an individual" (p. 63). In research on the SEM student, one dilemma consists of the fact that although students have disabilities or prerequisites to take into account, this contravenes the context and the students' experience: "difficulties experienced by children at school are best understood when the contexts in which children learn are examined along with learners' interactions within them" (p. 64). Here some researchers highlight *teachers' knowledge about support and the student* as cornerstones of work with SEM students (Bottge et al., 2007; Gal & Linchevski, 2010; Moscardini, 2010). Teachers' knowledge then includes knowledge about how to identify SEM students (Al-Hroub, 2010).

Conclusion

In this study we have investigated how the SEM student has been conceptualised in selected journals in mathematics education and special education research published during the period 2006–2013. Both building the framework and identifying journals and articles of importance were challenging tasks. The impact value is a tricky measure of value in the social sciences and, moreover, depends on how young the journal is and guidelines to authors. Owing to the interdisciplinary nature of the special education field, journals may very well be indexed under *development psychology* or *education* and therefore may not have been found by the index we used. The findings show that research writings, especially in the field of special education, have a categorical vocabulary. This was not expected, and it surprised us as professionals in the field of special education, given that awareness regarding the field's interdisciplinary challenges has been discussed by several scholars (e.g., Skrtic, 1995). There has also been a vivid debate on issues like inclusion and equity (Ahlberg, 2001; Goransson, Nilholm, & Karlsson, 2011; Nilholm & Alm, 2010; Skidmore, 2004) stemming from the Salamanca Declaration (Swedish Unesco Council, 2006). A striking fact is that very few articles explicitly discuss the conceptualisation of the SEM student. From 12 journals published during a 7-year period, we identified 28 articles using our procedure. There also seems to be ambivalence regarding the concept of the SEM student both within and between articles. The mathematical journals in general adopt a more relational perspective. In mathematics education, a social turn in research (Rodd, 2006; Lerman, 2000) might have contributed to this scenario, but it is also possible that the focus on the subject of mathematics draws research in this direction, whilst in special pedagogy the individual is in focus. We have adopted the concept *the student in need of special* *education in mathematics* in order to emphasise the social aspect. The word *in* is here of great importance. The student is *in* need, not *with* needs. Ambiguity regarding the very definition of the student in need became apparent in this study but is not surprising. There exists a view of research as a collective assignment taken on by individuals, and different fields and perspectives contribute differently to the definition. We do not believe consensus in the matter is desirable since fields complement one another, and the position of being an SEM student is complex and needs to be investigated from various perspectives. But, it is necessary to be explicit and systematic about the conceptualisation in order to avoid misunderstandings and misinterpretations. Otherwise follows a potential risk of badly coordinated and performed actions both in research and practice. Hence, a mission for further research is to investigate ways to develop more sustainable definitions of the SEM student. These definitions need to take both research and practice into consideration.

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An Instructional Design Perspective on Data-Modelling[1] for Learning Statistics and Modelling

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This theoretical and methodological paper discusses the emerging theoretical framing and methodological considerations in our efforts to develop a theoretical approach supporting instructional design for teaching and learning statistics and mathematical modelling. From an instructional design point of view aligned with the goals in governing curricula documents and real classroom constraints, we argue for the integration of the models and modelling perspective on teaching and learning mathematics with a data-modelling approach to facilitate students' learning statistics and mathematical modelling. An application of the framework is given and future research discussed.

Background

Statistics has been described as the science of models and modelling through which we make sense of the world using theory-driven interpretations of data (Shaughnessy, 1992). Indeed, the essence of statistical thinking is argued to be centred around developing, testing, interpreting, and revising models in order to understand our world and the diverse phenomena in it (Horvath & Lehrer, 1998). With this view of statistics, there are many parallels with the general view and on-going discussion on the use and role of mathematical modelling in the teaching and learning of mathematics. Both statistics and mathematical modelling have been put forward as increasingly more important for students to learn in order to cope with and be productive in their everyday and professional lives (in the case of modelling see Blum, Galbraith and Niss (2007); and, in the case of statistics see for example Gal (2002)). This suggests to us, as also hinted by English and Sriraman (2010), that statistics potentially provides a rich and productive venue for learning modelling on the one hand, but on the other, it also suggests that statistics may advantageously be learned through modelling. This interrelating connection between statistics and modelling (in a more general sense) constitutes an important strand in our present thinking and on-going work presented in this paper.

Following the definition used by Radford (2008) and Wedege (2010), we in this paper seek to develop a focused *theoretical approach*, that is, a framework "based on a system of basic theoretical principles combined with a methodology" (Wedege, 2010, p. 65). The sought theoretical approach should guide and support teachers and researchers in designing instructional sequences for the teaching and learning of statistics and mathematical modelling in everyday mathematics classrooms focusing on pre-defined learning objectives as prescribed in governing curricula documents. In addition to draw on the suggested benefitting symbiotic effects from intertwiningly learning statistics and modelling, we find it equally important for the theoretical approach we seek to develop to seriously acknowledge and respect the constraints affecting everyday teaching practices in schools. Two palpable constrains on teachers' practices are time constraints and content constrains, with the latter regulated in governing curricula documents specifying what it is that students should learn in the different grades and courses. Especially when the content and concepts becomes more advanced and abstract as the students become older and progress in their mathematics courses, the amount of time the teacher has available to spend on a given topic becomes considerably more limited. Ideally therefore, we want our theoretical approach to explicitly respect these constraints in that it should facilitate instructional design of productive learning situations for students that are focused from the point view of content as well as efficient.

To this end, we want to use the fundamental ideas underpinning the data-modelling approach for learning statistics described by Lehrer and colleagues (Horvath & Lehrer, 1998; Lehrer & Schauble, 2000; 2004) and to put these in the larger framework of model-development sequences (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003) within the models and modelling perspective on teaching and learning (Lesh & Doerr, 2003b). By adapting this integrating networking strategy (e.g. Prediger, Bikner-Ahsbahs, & Arzarello, 2008), we argue that the theoretical approach that emerges (1) provides a potentially highly productive symbiotic approach for learning statistics and mathematical modelling; (2) integrates prescribed curricula learning objective in a natural and time efficient way at all educational levels; and (3) results in instructional designs that provide promising learning possibilities for students and teachers as well as rich and productive research settings to further both the instructional designs themselves as well as the field of mathematics education research.

A modelling framework for instructional design of statistics learning

We now proceed to briefly discuss a models and modelling perspective on teaching and learning mathematics, before focusing on the ideas of Lehrer and colleagues' data-modelling approach. We end this section by presenting what should be considered the contribution of this paper: our emerging theoretical approach for instructional design of symbiotic teaching and learning of statistics and mathematical modelling.

A models and modelling perspective of teaching and learning

In a models and modelling perspective on teaching and learning (Lesh & Doerr, 2003b), a model is an externally representable conceptual system (consisting of objects, operations, relations, and interaction-governing rules) used to describe, explain, predict, or understand some other system (Lesh & Doerr, 2003a). In the case of mathematics, graphs, tables, algebraic expressions, computer animations, enacted actions, and spoke and written language are all examples of external representational systems.

From a models and modelling perspective learning is model development (ibid). Hence, we consider learning statistics to be for students to develop models for statistical reasoning. Here, the verb develop stresses, and refers to, the dynamic aspect of this process, where students' models repeatedly are developed, modified, extended and revised through "multiple cycles of interpretations, descriptions, conjectures, explanations and justifications that are iteratively refined an reconstructed by the learner" (Doerr & English, 2003, p. 112).

To support and facilitate students' development of models, sequences of structurally similar activities called model-development sequences have been introduced as tools for structuring teaching (Doerr & English, 2003; Lesh et al., 2003). A model-development sequence always begins with a model-eliciting activity with the purpose of putting the students in a meaningful situation where they are confronted with a need to develop or recall a model (c.f. (Freudenthal, 1983)). Other purposes of the model-eliciting activity are to make the students' previous experiences and models visible (to themselves, their peers and teachers) as well as explicit articulated objects that can be reflected upon and discussed.

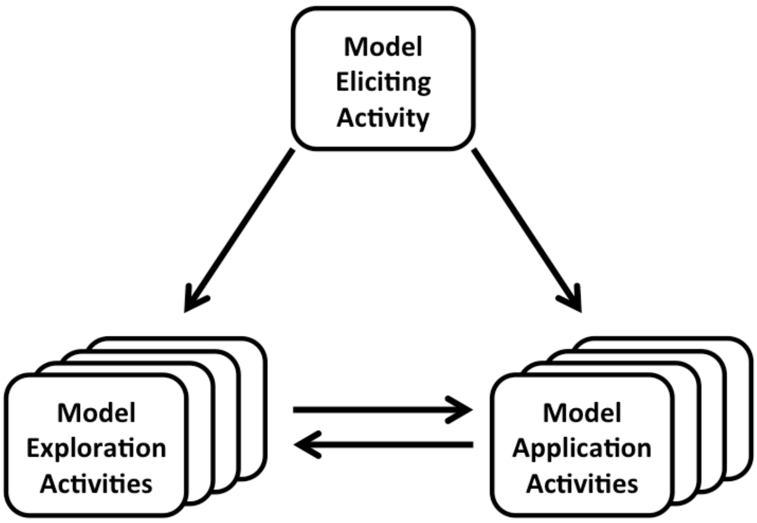

Figure 1. The general structure of a model-development sequence

In a model-development sequence, the model-eliciting activity is followed by one or more structurally related model-explorations activities and model-application activities (see Figure 1). The model-exploration activities focus on exposing and exploring the underlying mathematical structure of the elicited model. An important part here is for the students to work with different representations of the model as well as to develop an understanding for how to use different representations productively. The model-application activities allow the students to apply their models in other situations and contexts. Throughout working with the different activities within a model-development sequence students are constantly subjecting their (evolving) models to testing, modifying, revising and development.

Data-modelling

Lehrer and Schauble (2000) consider data-modelling to be a nested approach to students' classroom inquires with inherent processes facilitating development of students' ideas and models of big ideas and key concept in statistic. The idea is to put students in a, for them, realistic and meaningful "data-modelling context" (p. 636) where the starting point is student generated questions. Building on their understandings of the situation and problem at hand, the students develop and investigate feasible solutions (models) to their questions by engaging in a cyclic-like inquiry process illustrated in Figure 2. In other words, based on their contextual situated questions, the students have to identify and decide on what attributes are influencing the situation and are relevant for answering their questions; to collect data for these attributes (or select from given data); to choose ways of representing, organising, and displaying the data; and finally, to analyse the data and try to answer their questions by making inferences, often of an informal nature (English & Sriraman, 2010).

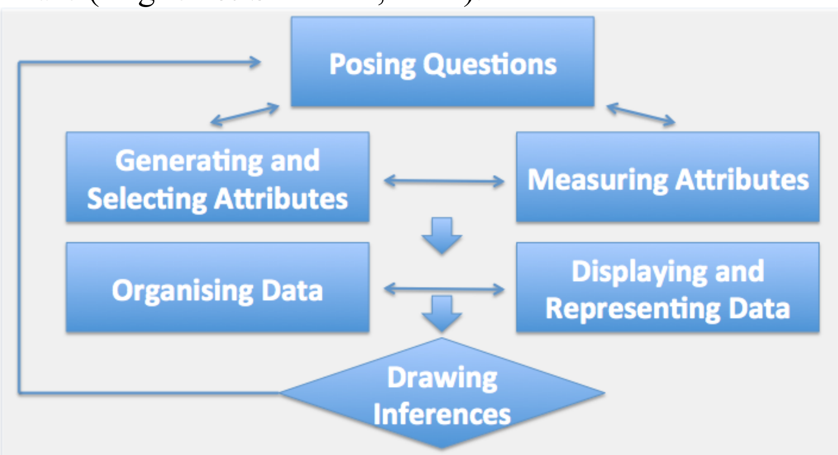

Figure 2. Components of data-modelling (Lehrer & Schauble, 2004) as adapted and presented by English and Sriraman (2010, p. 280).

Used with its open onset, drawing on "students-as-designer contexts for their fruitfulness in provoking and sustaining student engagement with data" (Lehrer & Romberg, 1996, p. 71), data-modelling in classroom settings normally spans over multiple lessons. From a learning point of view a data-modelling approach facilitates students' development of central and big ideas in statistics as well as giving them first hand experiences and a holistic view of the different components and aspects involved in statistical analysis (Lehrer & Schauble, 2000). From a research point of view on the other hand, studies involving implementing a data-modelling approach in everyday classrooms have proved to provide rich research sites for more focused studies, such as on student's development of understanding chance and uncertainty (Horvath & Lehrer, 1998) and students' development of understanding variation (Lehrer & Schauble, 2004).

A theoretical approach for instructional design for the teaching and learning of statistics and modelling – an extended data-modelling approach

We now turn to propose an expansion of the data-modelling approach discussed above. We do this by situating data-modelling as a model-exploration activity within a model-development sequence focusing on a particular learning objective by adapting an integrating networking strategy (e.g. Prediger et al., 2008). In doing this, we gain a structured way to focus on both learning and exploring specific statistics curricular content and mathematical modelling within a confined and limited number of activities.

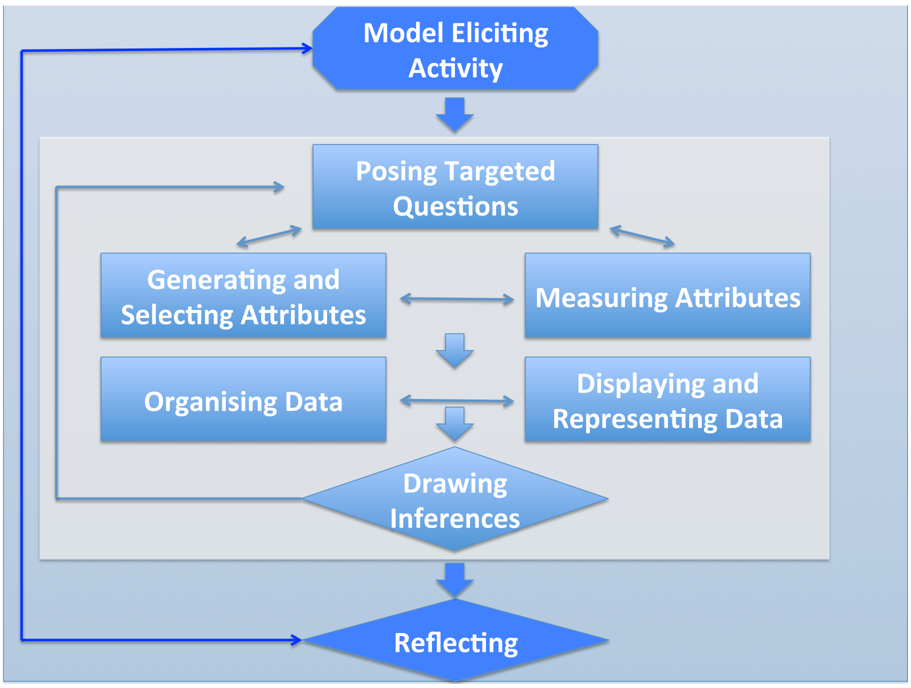

Figure 3. Components of the extended data-modelling approach

Situating data-modelling as a model-exploration activity within a model-development sequence means that in our extended data-modelling approach, an initiating model-eliciting activity must be added, that elicits the students' ideas and models with respect to the given learning objective (see Figure 3). Besides providing a situation were the students need to develop and express their thinking (models) about the targeted learning objective, this initiating modelling-eliciting activity also helps focuses the students in posing questions when engaged in the model-exploration activity (that is, in the original data-modelling approach). In other words, the model-eliciting activity funnel the students' thinking towards the particular learning objective in question, in that their experiences from the model-eliciting activity narrow down the viable questions for the sequential model-exploration activity and make the posing of questions more targeted. This is a shift from the original data-modelling approach represented by Figure 2, where the students' interests and own preferences to a large extent determine what questions to investigate and how to go about in trying to answer their posed questions, resulting in a more unpredictable activity in terms learning outcomes.

In our emerging theoretical approach (Figure 3) the initiating model-eliciting activity is followed by a traditional data-modelling activity (c.f. Figure 2) where students' content-wise focused elicited models are tested, challenged, adjusted, revised and developed as they investigate and analyse real data. It is the combination of these two first activities that lay the foundation for the students' symbiotic learning of statistics and mathematical modelling. In the theoretical aproach in Figure 3 there is a dialectic relationship between the goal to learn a specified statistical content and the goal to learn mathematical modelling. During the model-eliciting activity and the first model-exploration activity, the statistics' leaning objective is in the foreground whereas learning mathematical modelling is in background; one can say that the students learn statistical content using a mathematical modelling methodology, that mathematical modelling is the method and vehicle supporting the students' learning of statistics.

In addition to add an initial model-eliciting activity to the data-modelling approach, we also explicitly add an activity ending the model-exploration activity connecting back to the students' initially elicited models. By explicitly connecting back to the point of departure the students are offered an opportunity to reflect on both their own development as well as on the working methods and processes leading to it. This reflecting activity also reverses the emphasis in that the mathematical modelling process comes to the foreground and the focused statistical content in the background. To explicitly reflect and connect back to the starting point, or applied real world situation and original questions, is a key characteristics of all conceptualisations of modelling (Blum et al., 2007).

The extended data-modelling approach illustrated in Figure 3, can be seen as an extended model-eliciting activity in the original conceptualization of model development sequences by Lesh et al. (2003). That is, the models the students elicit and develop through engaging in this extended model-eliciting activity can then be further developed, explored and applied though carefully sequenced model-exploration activities and model-application activates.

By virtue of design, the theoretical approach in Figure 3 offers and allows an opportunity to design instruction that simultaneously integrates teaching and learning of focused content learning alongside an additional more general learning object or more general abilities such as problem solving, getting a holistic view of mathematical modelling, and for the students to experience the uses, benefits and power of mathematics.

An example and application of the theoretical approach

We now continue to give an example illustrating our emerging theoretical approach in action. Due to the limited space available we will only focus on some selected design decisions of the actually designs informed by the theoretical approach developed in this paper, and not present any data, analysis or results. The example is from on-going work investigating how students develop their inferential reasoning.

Developing students' informal statistical inference reasoning

Recent research in statistics education has focused on different aspects of the role and function of students' informal statistical reasoning in their everyday lives and for learning formal statistical inference (Biehler & Pratt, 2012; Makar & Rubin, 2009). In this context, the theoretical approach presented above was used to design a teaching experiment to support students developing their informal statistical inference reasoning. Of more specific interest was students' informal statistical inferences drawn from samples to larger populations (statistics) and what attribute students consider to be relevant for making such an inference legitimate (modelling).

The teaching experiment was situated in the context around the growing concern of obesity and health issues due to peoples sedentary and immobility, and the initial model-eliciting activity the student worked on in small groups was:

> Research suggest that young people should walk at least 10 000 steps a day to stay physically fit. What is the probability that a person of your age completes more that 10 000 steps a day? Your task is to write a letter to the teacher where you present assumptions, reasoning and calculation you have done to solve the task.

The students drew on their previous knowledge and experiences to identify relevant attributes for setting up a model as well as to estimate numerical parameters in order to come up with a probability. By engaging in this model-eliciting activity the students' ideas (models) became explicit objects of thoughts, and open for discussion. The activity also allows the students the opportunity to make visible what mathematical and statistical knowledge and constructs the considered applicable and relevant for the situation.

In the data-modelling task that followed, the students used pedometers to collect empirical data of their own physical activities. Based on the data collected by the members of their group, the students then revisited the question from the initial model-eliciting activity. The variation of number of steps in the collected data supported the students in verifying the relevance of their prior identified attributes or in identifying new ones.

The reflecting activity that ended the teaching experiment was two-folded; partly it consisted of the students writing a report, and partly is consisted of a teacher-lead whole class discussion. Both forms of reflections focused on the informal statistical inferences drawn by the students as well as the modelling processes leading up to their conclusions. In addition, during the whole class discussion, all the students' samples from their different groups were aggregated, which was brought issues of sample size and numbers of samples on the table. Here, these ideas about the role of sample size and the numbers of sample in making inferences suggest directions for further model development and sequential model-exploration activities as well as model-application activities. This latter point however, is subject for future research.

Conclusions, implications and future research

In this paper we have presented an emerging theoretical approach for supporting instructional design for teaching and learning statistics and mathematical modelling symbiotically. We drew on fundamental ideas underpinning the data-modelling approach for learning statistics described by Lehrer and colleagues (Horvath & Lehrer, 1998; Lehrer & Schauble, 2000; 2004) and integrated (c.f. Prediger et al., 2008),. these into the larger framework of model-development sequences (Lesh et al., 2003) within the models and modelling perspective on teaching and learning (Lesh & Doerr, 2003b). In addition, we sought to integrate sensitivity to constraints imposed on everyday teaching practices from the limited amount of time available and prescribed student learning outcomes in governing curricula documents.

By drawing on well establish research methodologies and approaches, we argue that our emerging theoretical approach in a productive way pulls together fundamental theoretical ideas and respects important real classroom constraints resulting in a supportive model for thinking about, designing and developing teaching and learning of statistics and mathematical modelling symbiotically. In addition, since the definition of models within the models and model perspective is broad and general, the emphasise on prescribed curricula learning objectives facilitate a natural as well as time efficient and flexible approach for instructional design that arguably could be applied at all educational levels.

We acknowledge that that there are challenges in adapting and implementing teaching based on the emerging theoretical approach presented in this paper. However, as well as potentially resulting in instructional designs that provide promising learning possibilities for students and teachers, we plan to use a design based research paradigm (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) to continue to develop our theoretical approach. In this context our work presented here provides us with rich and productive research settings to further both the instructional designs themselves as well as the field of mathematics education research.

A line of research that we find specifically fruitful and appealing, is to further draw on the related literature discussing the different components of our theoretical approach combined with empirical investigations to develop a set of design principles specific for teaching and leaning of statistics and mathematical modelling symbiotically. In the case of model-eliciting there are six well established design principles (Lesh, Hoover, Hole, Kelly, & Post, 2000) which provide a natural starting point for this endeavour. In addition, we intend to build and continue to further the work on the emerging design principles for model-exploration activities and model-application activities initiated in Ärlebäck and Doerr (submitted).

Notes

1. We acknowledge that much of the referred literature in this paper is American and hence uses the America spelling *modeling*. Also note that at times the notions *data modeling* and *data-modeling* is used interchangeably in the literature. We however, for consistency reasons, consequently use the English spelling *modelling* and the notation *data-modelling*.

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Brackets and the Structure Sense

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Brackets are essential structure elements in mathematics. However, students have shown to have scattered understanding of the concept of brackets and how they are used in mathematical expressions. In this paper we present data that illustrate students' perceptions of the word "brackets" and how these perceptions influence their use of brackets in numerical expressions. Based on our data we argue that the teaching of the concept of brackets also need to describe brackets as ordered pairs where each symbol has a unique counterpart and that insertion of brackets can, but does not have to, modify the structure of an expression.

Introduction

Students' understanding and misunderstanding of letters in and the structure of algebraic expressions has since long been well described (Küchemann, 1978; Rosnick, 1981; Kieran, 1989). A central set of symbols for the algebraic structure is the brackets. Brackets constitute an essential part of algebra and distinguish, together with rules for the order of operations, the algebraic language from spoken everyday language (Freudenthal, 1973, p. 305). However, students' understanding of the bracket symbols is not equally well documented in mathematics education research.

Typically the concept of brackets is taught alongside with rules for the order of operations. Brackets and their properties are often introduced to students in a single sentence saying that "brackets show what should be calculated first". However, this is not necessary always true. Two examples; in the expression $4 + 5 - (2 + 3)$ one could very well add 4 and 5 before adding the 2 and 3 within brackets, and when solving the equation $(x + 3) \cdot 2 = 8$ the first operation is not to calculate what is inside the bracket but to divide the equation by 2.

In addition, there are misconceptions of the word and the concept of brackets, some known and described in literature. It has been shown that students can interpret "brackets should be calculated first" as "brackets should appear first" in a left-to-right meaning (Kieran, 1979). In addition, brackets can, when used as a marker for negative numbers as is common in the Swedish mathematics teaching tradition, cause confusion to what should be calculated first (Kilhamn, 2012). As an example, what should be calculated first in (-2) – a negative two?

Moreover, Hewitt (2005) has shown that he word “brackets” often is translated literally into a mathematical expression – ignoring the structure of the expression to be written. He also showed that the word “bracket” appears ambiguous, as seen when students read out equations loud or translate text to equations.

Another possible cause of problem is that brackets are used with different purposes in mathematical expressions. Brackets can be used to emphasise the intended order of operation but otherwise be mathematically useless, like in $\frac{1}{(x+1)}$, or brackets can be necessary parts of the expression which without them would have another meaning, like in $2 \cdot (4+3)$. Linchevski and Livneh (1999) have suggested to use emphasising brackets in $a \pm b \cdot c$ type of expressions in order to detach the number (b) from the operation ($\pm$), supporting the learning of a structure sense. Useless, emphasising, brackets can indeed help students see algebraic structure (Hoch & Dreyfus, 2004), and emphasising brackets can increase success rates in arithmetic expressions (Marchini & Papadopoulos, 2011). But one has to be careful when using emphasising brackets as it has been shown that they may impede the learning of precedence rules (Gunnarsson, Hernell & Sönnerhed, 2012). Overall, there are plenty of reasons to look deeper into the teaching and learning of bracket symbols.

Aim and scope of the study

The aim of the study discussed in this paper is to analyse students’ perception of mathematical brackets. We would like to achieve this by answering the following research question: How do students perceive the word “bracket” and the concept of brackets in mathematical expressions?

Description of the study

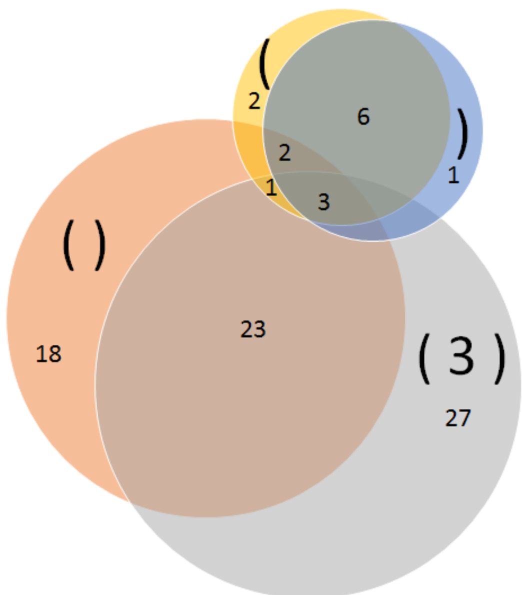
For this study 84 students, aged 14-15 (school year 8), in eight different classes in four different Swedish schools participated in a paper-and-pencil questionnaire. The questionnaire contained ten tasks each including one or more expressions to evaluate. Each student was asked to evaluate in total 35 different arithmetic expressions, a few of them will be discussed in this paper. Details of the full questionnaire can be found in (Karlsson, 2011). No calculators were allowed during the test.

In the Swedish teaching tradition students typically first meet brackets in the seventh grade. By involving eighth grade students we therefore probed the students’ perceptions and their use of brackets in their initial phases of learning the concept, but they should have met brackets in their mathematics teaching at least the year before. The schools and classes were not selected by any statistical method, but had a reasonable distribution regarding gender, ethnicity and social background.

The data was analysed mainly by qualitative methods. However, to some degree the data were also quantitatively summarised. Though the main analysis was made by categorising the different perceptions that became evident in the students' answers. The focus in this brief report is not on the analysis of single students' different answers, but on describing the different sets of misunderstandings that came up in this study. The answers to the different tasks were therefore analysed (by categorisation) and the perceptions found were cross-correlated between different tasks. Hence, the categorisation system is not in focus in this paper, but rather the outcome of the cross-correlation between different tasks.

Results

The students were asked to choose what they perceived as a bracket. The actual question that was asked was "Which, or which one of the following is example of a bracket?" with the alternatives "(", ")", "( )" and "(3)" and with tick-boxes for each type. Figure 1 shows a Venn-like diagram of the distribution of student answers to this question. The numbers in the diagram in Fig. 1 show the number of students ticking each separate box. The majority of the students considered the empty pair of brackets and the brackets with a content to be exam-



4. Vilket eller vilka av följande visar exempel på en parentes? Markera med ett kryss:

☐ (☐ ( )
☐) ☐ (3)

Figure 1: The space of all answers to the question "Which, or which one, of the following shows example of a bracket?"

ples of brackets. The largest group (27 students) marked only the alternative "(3)", a bracket with content, as an example of a bracket, whereas the second largest group (23 students) marked both "( )" and "(3)", but not a single left "(" or right ")" symbol. A small number of students (2+6+1, i.e. in total 9 students) answered that a single symbol, either a left-handed or a right-handed symbol or both – but not in combination, represent a bracket. An even smaller number of students (1+2+3, in total 6) marked both single symbols and symbols in combinations to be examples of brackets.

Figure 2 shows a few students' answers to which brackets that are considered unnecessary in a number of different expressions. In Figure 2(a) the question with the complete set of mathematical expressions is shown. The student has in this case marked all unnecessary brackets except the ones that emphasise the precedence of multiplication over addition/subtraction. Two examples of answers where the marked bracket symbols are not corresponding to a conventional pair are shown in Fig. 2(b)-(c).

In Figure 2(d) only a single bracket symbol is marked as unnecessary. The closing bracket in the midst of the expression appears to be considered as necessary. In the answer in Figure 2(e) it appears as if the student considers multiple brackets unnecessary, i.e. that it should be sufficient with a single bracket symbol. Almost the same kind of perception of brackets is shown in the answer shown in Figure 2(f), where outer multiple brackets have been deemed unnecessary.

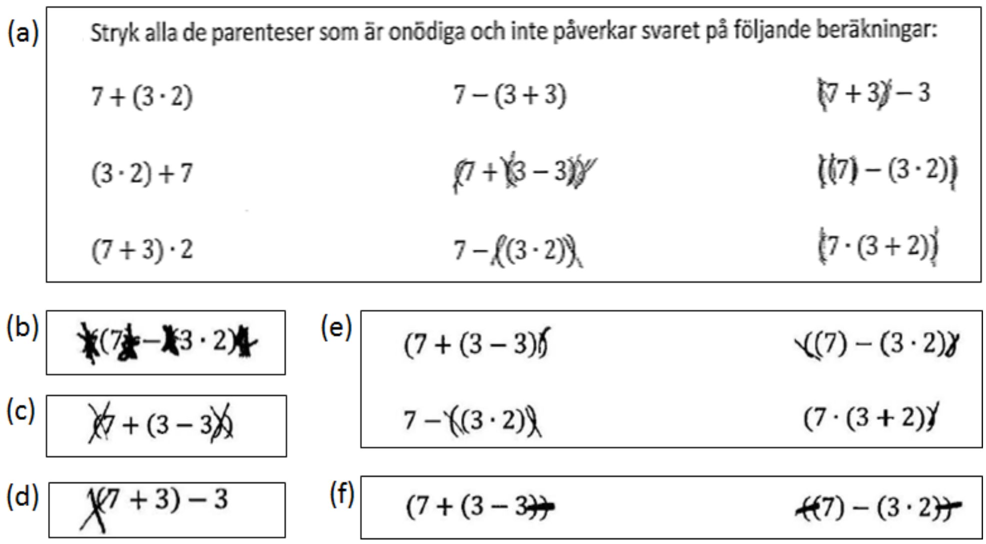
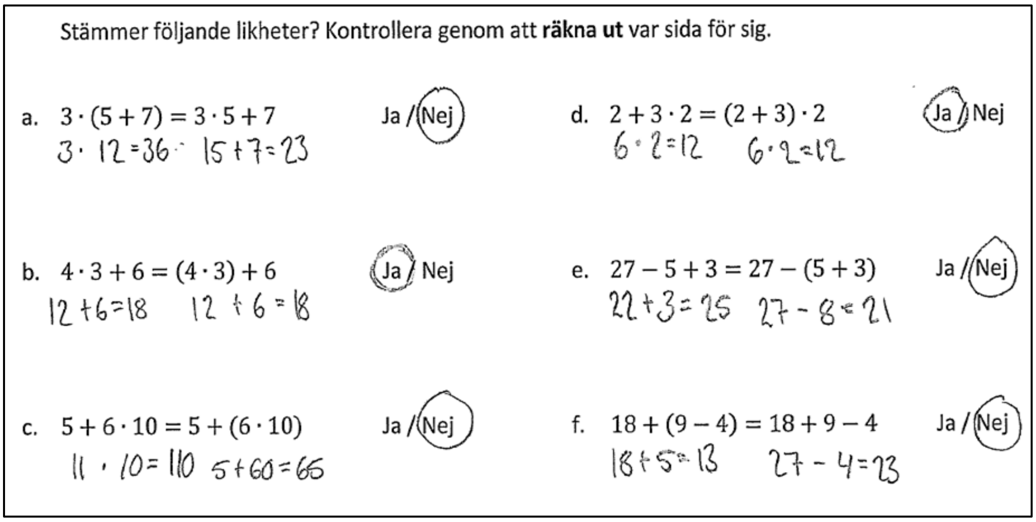


Figure 2: A selection of answers to the task "Cross out all the brackets that are unnecessary and do not affect the answer to the following calculation".


Stämmer följande likheter? Kontrollera genom att **räkna ut** var sida för sig.

a. $3 \cdot (5+7) = 3 \cdot 5 + 7$ Ja / Nej
3 · 12 = 36 · 15 + 7 = 23

d. $2 + 3 \cdot 2 = (2+3) \cdot 2$ Ja / Nej
6 · 2 = 12 6 · 2 = 12

b. $4 \cdot 3 + 6 = (4 \cdot 3) + 6$ Ja / Nej
12 + 6 = 18 12 + 6 = 18

e. $27 - 5 + 3 = 27 - (5+3)$ Ja / Nej
22 + 3 = 25 27 − 8 = 21

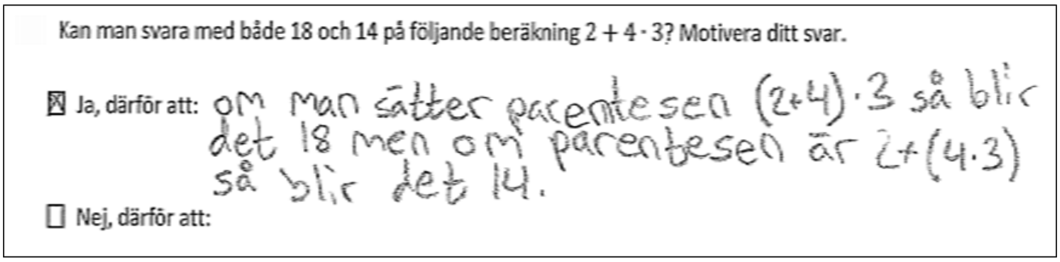
c. $5 + 6 \cdot 10 = 5 + (6 \cdot 10)$ Ja / Nej
11 · 10 = 110 5 + 60 = 66

f. $18 + (9-4) = 18 + 9 - 4$ Ja / Nej
18 + 5 = 13 27 − 4 = 23

Figure 3: One student's answer to the question "Are the following equalities correct? (Yes/No) Verify by **calculating** each side separately".

The students were also asked to evaluate numerical equations, see Figure 3. In this task all six equations contained brackets on the left or the right hand side. In three expressions the brackets were mathematically useless and in the other three the brackets were necessary in order to maintain the structure of the expression. In the student answer shown in Figure 3 it appears as if the student regards brackets to signal precedence, but that without brackets the expression should be evaluated from left to right. Consequently the student answers that, e.g., $5 + 6 \cdot 10$ should be evaluated differently than $5 + (6 \cdot 10)$, and that $(2 + 3) \cdot 2 = 6 \cdot 2$ [*sic*!] is the same as $2 + 3 \cdot 2$.

Another student answers the question whether the expression $2 + 4 \cdot 3$ is ambiguous, with "[yes, because:] if you put the bracket $(2 + 4) \cdot 3$ it will be 18 but if the bracket is $2 + (4 \cdot 3)$ it will be 14", see Figure 4. Hence, this particular student has answered that the evaluation of the expression depends on where you put the brackets. But this is not a single student phenomenon. A frequent answer to this question on the questionnaire was "yes" (23 students). However, among the other answers there were 5 blanks/don't know and a small number of motivations like "[no, because:] there could only be one answer". The student answer shown in Figure 4 is one of those revealing a perception of brackets as if they could be used arbitrarily. But also in the "no"-responses there were indications of alternative perceptions of brackets as in e.g. the answer "[no, because:] there are no brackets and then it must be 18". This latter can be seen as yet another example of when the absence of brackets leads to a left-to-right calculation.


Kan man svara med både 18 och 14 på följande beräkning $2 + 4 \cdot 3$? Motivera ditt svar.

☒ Ja, därför att: om man sätter parentesen $(2+4)\cdot 3$ så blir det 18 men om parentesen är $2+(4\cdot 3)$ så blir det 14.

☐ Nej, därför att:

Figure 4: One student's answer to the question "Is it possible to answer both 18 and 14 to the following calculation $2 + 4 \cdot 3$? Motivate your answer. (Yes, because:/No, because:)".

Discussion

Even though the students have been introduced to brackets there is still a wide spectrum of misconceptions that can be seen in the data. We cannot exclude, actually we find it very likely, that students' preconception of the word bracket plays a major role to this. Possibly, everyday communication where single bracket symbols are frequently used as, e.g., in "smileys" :-) could have an influence on the perception of brackets as a single left or right arch. We also note that the Swedish language is ambiguous regarding the use of the word "parentes". In the official Swedish language the word *parentes* refers to an inserted expression ("inskjutet uttryck") according to the Swedish Academy glossary (The Swedish Academy, 2006), and a single bracket should be called "parentestecken". The equivalents of "opening bracket" and "closing bracket" are used but are often called "start parentes" and "slut parentes".

In addition, the phrase *within brackets* (note "bracket" in plural) would in Swedish be translated to *inom parentes* (singular). Hence, we anticipate that the students' language could be a source for some misconceptions observed in our data. One could argue therefore that this is a local problem, but as we see that problems regarding students' ways of handling brackets in mathematical expressions appears also in the English language (Kieran, 1979; Hewitt, 2005) we believe there are more general implications.

Students' perception of the word "brackets"

The alternatives the student could choose from in the question in Figure 1 were fixed and no openings for alternatives were offered. The options to mark were single left symbol "(", single right symbol ")", an empty pair of symbols "( )" and a pair of symbols with some content "(3)". Other alternatives could be possible, but we believe that the perceptions of brackets are mainly revealed in *how* the brackets are used in mathematical expressions, described in next section.

It is interesting to note, however, that only a small number of students do consider both single symbols and paired symbols to be examples of brackets. The majority considers a bracket to have a content (or possibly that it *can* have a content). This group represent 27 (+23), as shown in Figure 1. This is consistent with the viewpoint that in the Swedish language the word brackets ("*en parentes*") represents an inserted expression, i.e. the content within a pair of brackets.

Students' perception of the concept of brackets

The different ways of perceiving brackets, as single symbols, as empty pairs or as pairs with contents can lead to problems when translating text to an algebraic expression, as shown by Hewitt (2005). In the full questionnaire (but not shown in this paper) we also included a similar task, and we find the word "bracket" interpreted as *single symbols* or as *empty pairs* or *pairs with contents* – the same categories as in the perception of the word brackets discussed above. However, we also find, in agreement with Hewitt (2005), that students do not consider the structure of an expression when translating words to symbols. A substantial part of the students does not seem to consider the structural properties of bracket symbols.

Even when the brackets are perceived as a pair they do not necessarily have to be perceived as ordered pairs. In Figure 2(b-c) we find examples where it appears as if the students are forming the bracket pairs somewhat arbitrarily. If we recreate the answer in Figure 2(b) the student have left the rounded brackets and crossed out the square brackets in this expression $[(7] - [3 \cdot 2)]$. Of course in this expression all brackets can be considered superfluous. The student does seem to acknowledge that brackets appears in pairs. But we focus on the new pair that the student forms. This is not a pair in the sense that a particular opening bracket has a corresponding closing bracket. What this student seems to have missed is that brackets appear in *ordered* pairs.

The same question also revealed another misconception. In Figure 2(e) a student answer is shown where multiple bracket symbols appear to be perceived as unnecessary. The student seems to consider $(7 \cdot (3 + 2))$ to be the same as $(7 \cdot (3 + 2)$. In this case the two opening brackets are considered to share the same closing bracket. It appears as if the student has missed that for every opening bracket there exists one *unique* closing bracket, and vice versa. Possibly this could also be true for the student giving the answer in Figure 2(f). But that answer could also be related to the answer in Figure 2(d). In this case it appears as if the brackets are perceived as only separating inner parts of an expression. The student in this example suggests that after removing the unnecessary brackets in the expression $(7 + 3) - 3$, what should be left is $7 + 3) - 3$. We believe this is an example of where brackets are considered to be single symbols, not pairs.

In conclusion our data suggest that brackets are perceived as single symbols, empty pairs of pairs with content. The pairs can by students be perceived as being formed by any two combinations of single bracket symbols and need to even be perceived as an even number of single symbols (e.g. when two "left brackets" are paired with one "right bracket").

Students' use of brackets as part of mathematical notation

Our data support the observation by Kieran (1979) that brackets are a signal of what should be calculated first. Even though it is not reported here we do also see examples in our data where students move brackets to the left to "do them first". However, the data also show examples of when the lack of brackets is taken as a signal that the rules for the order of operations do not apply. In Figure 3 we see such an example where brackets are used as necessary parts of the structure of the expression. But this example also reveals that in the lack of brackets the structure is considered different, i.e. left-to-right instead of precedence. The student seems to have missed that *brackets show the structure* of an expression. This appears also to be true for the student whose answer is shown in Figure 4. This student seems to have missed the information that *brackets cannot be inserted arbitrarily* without changing the structure of the expression (and the result of the calculation). We believe this shows that it has to be made clear that there is a close connection between the structure of a mathematical expression and where in the expression brackets can be inserted without distorting it.

In conclusion, we find that students do not necessarily perceive brackets as the important structure element described by Freudenthal (1973, p. 305). Brackets can be perceived as a signal to use the precedence rules, but without the brackets the expressions could be evaluated left-to-right. Brackets can be perceived as something that can arbitrarily be inserted into an expression.

Structure sense, brackets and educational implications

The term *structure sense* was coined by Linchevski and Livneh (1999) in order to describe difficulties in algebra based on lack of understanding of structure of arithmetic expressions. We believe that in order to fully understand the mathematical structure it is necessary to also, or possibly first, understand how terms are grouped and how different operations work together. But grouping of terms cannot be made arbitrarily. Hence, when teaching mathematical rules for the order of operations, emphasising brackets can be used. This is analogous to the use of emphasising brackets by Hoch and Dreyfus (2004) and Marchini and Papadopoulos (2011).

However, as supported by our data the present introduction of brackets appears to be insufficient. We therefore believe that the introduction of brackets needs to emphasise the properties of brackets, not just their place in the rules for the order of operations. Particularly, based on our data, we suggest that brackets are presented as ordered pairs where each bracket symbol has a unique counterpart. That the insertion of brackets is shown to be able to change the structure of an expression, but that brackets not necessarily have to induce such a change.

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Analysing Instrumental and Pedagogic Situations in Preschools using the Didaktic Space

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Researchers rarely discuss methodological issues in regard to preschool mathematics education and if they do, they do not take their starting point from reconceptualisations of what mathematics might be for preschool children. This paper presents as an analytical tool the "didaktic space" that arose when responding to issues related to the analysis of data collected in a Swedish preschool. The issues that arose from categorising situations using Bishop's six activities required some reconsiderations of the methodology in relationship to the research questions. The paper discusses how methodological decisions can affect the analysis and the future possibilities that the didaktic space offers.

Methodological issues in understanding preschool mathematics

Clements and Sarama (2007) in reviewing literature on preschool mathematics education research identified the different theories (empiricism, neo-nativism and interactionism) that have been used to discuss how children learn and use mathematics—"mathematical ideas are represented intuitively, then with language, then metacognitively, with the last indicating that the child possesses an understanding of the topic and can access and operate on those understandings" (p. 464). However, most of the research that they reviewed assumed that preschool mathematics can only be understood in relationship to school mathematics. Such a starting point is problematic because of the many differences between school and preschool.

School students encounter mathematics in mathematics lessons and associated homework, that is, in situations clearly labelled and demarcated as mathematical. Therefore, it is possible to define school mathematics as a social practice as defined by Fairclough (2003):

> Social practices can be thought of as ways of controlling the selection of certain structural possibilities and the exclusion of others, and the retention of these selections over time, in particular areas of social life. (p. 23)

The same students might engage in mathematical activities in out-of-school situations; however, these are embedded in social practices very different to school mathematics (Lave, 1988), and are likely, in Bernstein's (1971) terms, to be more weakly classified and framed. The poor compatibility between the social practices of school mathematics and every-day life is one reason that every-day situations typically are not experienced as being connected to school mathematics.

With its long institutional and pedagogical tradition as an institution for the care and upbringing of young children (Roth, 2011), Swedish preschools also can be considered a social practice into which children are enculturated (Bishop, 1988) or become participants (Wenger, 1998). In learning terms, children learn to do preschool in the same sense that school students learn to do school. In a Swedish preschool setting with its strong tradition for perceiving children as learning through play, clearly demarcated situations include "fruit time", "circle time", "play" indoors or outdoors and others but not situations labelled "lesson" as is typical in a school setting, even though, in some preschools, situations labelled "mathematics" may occur.

Therefore, it is more difficult to attach the label of social practice to Swedish preschool mathematics. Whereas school mathematics is strongly classified and framed (Bernstein, 1971), thus making it easily recognisable as a social practice, this does not seem to be the case for mathematics in Swedish preschools as it is not delineated sufficiently to qualify as a social practice. Looking for mathematical activity characteristic of the social practice of school mathematics may give few results, and may not be appropriate, given the very different curricula (Skolverket, 2010; 2011). This raises the question of how to identify children – and preschool teachers – as being involved in mathematics, whatever that might be in preschools, even when they are not aware of it, as well of the meaningfulness of such identification. A research frame set by school mathematics could lead to the question of "where is the (school) mathematics" rather than, "in what ways are preschool children engaged in which mathematical activities". Therefore, the possibility for understanding the breadth of the mathematical activity in which children engage at preschool is reduced if we limit ourselves to only look for mathematical activity in situations labelled as such and in which all participants are aware of the label.

As a result of identifying the problem with viewing preschool mathematics only in relationship to school mathematics, we chose in our previous work (Johansson, Lange, Meaney, Riesbeck, & Wernberg, 2012; Helenius, Johansson, Lange, Meaney, Riesbeck, & Wernberg, 2014 this volume) to consider preschool mathematics as one version of Bishop's (1988) 6 mathematical activities – discussed in the next section. Here it suffices to say that this decision has required us to reflect more widely about issues that emanated from this choice, such as who is doing the classification and for what purpose. In this paper, we discuss some of these issues in relationship to our research question "in what ways are preschool children engaged in which mathematical activities".

Mathematical activities

In his book, Bishop (1988)

> presented the case that six key 'universal' activities are the foundations for the development of mathematics in culture. … All cultures have necessarily developed their own symbolic technology of mathematics in response to the 'demands' of the environment as experienced through these activities. (p. 59)

The mathematical activities were Counting, Measuring, Locating, Designing, Playing and Explaining, which respectively, and in short, were answering questions involving quantification (how many? how much?); space and shape (where? what?); abstraction, hypothetical thinking and reasoning (how to? why?). According to Bishop, these activities are present in all cultures, albeit, in different forms depending on the particular social and environmental needs. He referred to the "internationalised discipline of mathematics" (p. 57) as Mathematics with a capital M and saw it as one "version" of the 6 activities. In the cases of Mathematics and school mathematics, the 6 mathematical activities are 'solidified' into distinct social practices (Fairclough, 2003). Seeing academic and school mathematics as social practices resonates with Bishop's conceptualisation of mathematics as a cultural activity. Both perspectives highlight mathematics as a human activity, which, rather than being one intellectual, non-material or even trans-human edifice, comprises a range of socially and culturally situated practices, each of which is characterised by a set of sayings, doings and relatings (Kemmis & Grootenboer, 2008) that affords and attributes purpose and meaningfulness to the activity.

As indicated earlier, we chose Bishop's 6 mathematical activities to be the "spectacles" with which to look for mathematics in preschools. We could identify all of Bishop' 6 mathematical activities in situations that were video-recorded in a Swedish preschool in 2011 (Johansson et al., 2012). One consequence of Bishop's conception of mathematical activities as embedded in cultural and, hence, social practices, is that the mathematical activity in a situation does not depend on it being recognised by the participants. It is sufficient that the situation is recognised as involving a mathematical activity by the researchers.

Yet, the classification was not straight forward. Unlike MacMillan (1998) who also had used Bishop's 6 activities in preschool mathematics education research, in any one situation, we often could identify more than one mathematical activity. Although a practical challenge, this did not require any rethinking about the classification.

In contrast, while doing the classification it became clear that we needed to consider the role the mathematical activity had in the situation. Sometimes it seemed to be the focus of the situation and at other times, it seemed to be an unrecognised tool for resolving a problem. An example of the first is where the teacher drew attention to the shape of leaves, collected by children because they liked collecting them. By highlighting the mathematics, the teachers turned the focus of the situation away from collecting leaves and on to the mathematical activity Designing. An example of the second type could be a situation where a child had filled a bucket of sand in order to make a sand castle. However, the bucket was too heavy to be turned over and so no castle could be produced. In this case, the child had to work out that to turn the bucket it had to be less heavy. This required the amount of sand in the bucket to be reduced, which was done by scooping some sand out. The mathematical activity Measuring was involved in solving the problem but was not the focus, or centre of awareness.

While the notion of mathematics as being comprised of 6 mathematical activities resolved one methodological problem, that of identifying the mathematics of preschool, it raised another issue. This required some rethinking because it seemed that the two different purposes did provide more details about how to answer our research question "in what ways are preschool children engaged in which mathematical activities".

Instrumental and pedagogic situations

We needed to find some way to discuss the different purposes and the affect that they had in responding to our research question. Subsequently, we chose to use Walkerdine's (1988) distinction between instrumental and pedagogic tasks.

> This classification used the designations *instrumental* and *pedagogic* to describe certain kinds of tasks at home and was a distinction originally devised in relation to practices involving *number* in the home. Instrumental referred to tasks in which the main focus and goal of the task was a practical accomplishment and in which numbers were an incidental feature of the task, for example in cake-making, in which the number *two* might feature in relation to the number of eggs needed and so on. In the pedagogic tasks numbers featured in a quite different way: that is, numbers were the explicit focus of the task. On such occasions the focus was predominantly the teaching and practice of counting. So, for example, a child might be asked to count her coat buttons for no other purpose than to practise the count. (Walkerdine, 1988, p. 81; italics in the original)

However, when Walkerdine tried to use the classification on parent–child interactions involving size relations, it was not so easy:

> I found the exercise difficult. The usages did not always seem mutually exclusive and I was not convinced by my own categorisation. In addition, there appeared to be some exchanges that did not fit either of the classifications. In these exchanges the mother appeared to be *commenting* on an activity or on something which had been done or seen. In these cases the mother did not appear to be instrumental, in that the exchange was not actually part of a practical activity, but then neither was the purpose explicitly didactic. (Walkerdine, 1988, p. 86; italics in the original)

Hence, it could be that the designations instrumental and pedagogic was more suitable to classifying situations involving the mathematical activity Counting, maybe because features of the school mathematics version of Counting (counting, doing sums, practicing multiplication tables) figure so strongly in the public discourse about mathematics. In our video recordings mentioned earlier, we succeeded in finding instrumental and pedagogic situations for each of the six mathematical activities (Johansson et al., 2012), thus suggesting that the first of the issues raised by Walkerdine was not relevant in relationship to our data set. We also did not have examples of the commenting that Walkerdine identified, perhaps because teachers in preschool settings are more likely to engage with children in a situation rather than just comment about what was going on.

However, another methodological issue did arise. This was one of perspective, that is, whose perspective of the situation was adopted in the analysis? Although our original assumption was that using Bishop's 6 activities would mean that the classification based on our researcher's gaze was appropriate, our reflections now made it clear that such an assumption was naive. Situations could be classified as either instrumental or pedagogical but would not necessarily appear the same to the participating children and teachers. In some cases, it seemed that a situation could be instrumental for the child but pedagogic for the teacher. In the how-to-turn-the-bucket-over situation, the child was engaged in the practical accomplishment of making a sand castle. Hence, the mathematical activity in the situation was instrumental for the child (i.e. IC). The teacher, watching the child, seemed to recognise the child's problem and supported the child working out the solution (taking out sand) by verbalising her interpretation of the child's tacit reasoning. Hence, it appeared to be a pedagogic situation for the teacher (PT), in which she supported the child's engagement with the mathematical activities of Explaining and Measuring. If she had just told the child to take out sand or done it herself, then we would have classified it as an instrumental situation for the teacher (IT).

From our reflections on the methodological issue of whose perspective, we decided to change our conceptions of situations being either pedagogical or instrumental to a classification that would allow for a more nuanced interpretation. Consequently, we decided to situate the classification of situations in a two dimensional grid with the axes instrumental–pedagogic for the child(ren) respectively for the teacher (Figure 1). According to the analysis above, the how-to-turn-the-bucket-over situation would be located in quadrant ❹ (IC–PT). If the teacher had taken the sand out herself, it would be in quadrant ❶ (IC–IT). The grid spans a field of didaktic affordances and we hence label it “didaktic space”.

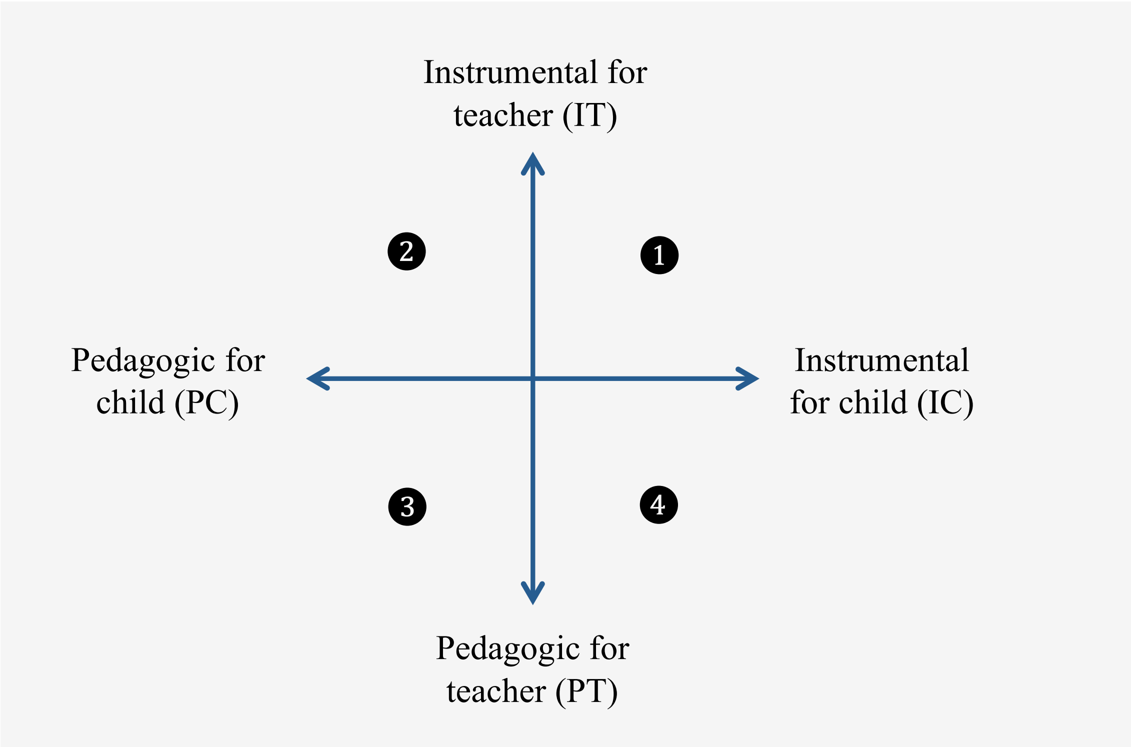

Figure 1. Didaktic space. The numbers refer to the quadrants.

The didaktic space

Each of the four quadrants in the didaktic space represents situations with a particular didaktic makeup. Situations would be located in quadrant one, when the teacher and the child(ren) were solving a problem, involving one or more mathematical activities. In these situations, none of the participants expected to teach or learn anything. Although one participant may be more knowledgeable about how to solve the issue the focus for all participants is on the resolution of the problem, not on the process of resolution, which opens up possibilities for teaching.

In quadrant two, the teacher may be focused on solving a problem whereas the child(ren) would be focused on teaching the teacher or themselves about some aspect of a mathematical activity. Although there were few examples of this in the situations in our data set (Johansson et al., 2012), in other data sets, it is possible to imagine a situation in which a teacher is focused on packing up materials, while the child is focused on learning about different attributes while doing it.

In the third quadrant, the focus for both the teacher and the child(ren) is on teaching/learning about a mathematical activity. Usually, the teacher is the one who has the role of teacher and the child(ren) the role of learner. However, there is a potential for the roles to be reversed. The teaching may just involve making children aware of a specific feature or more formally requiring a child to pay attention to and learn the material in a way that the teacher can recognise. The PC–TP combination is characteristic of school mathematics. The gain from it is a strict focus on content, on "mathematics" but the loss may be the motivation and purpose of engaging in the mathematical activity.

In the final quadrant, the teacher's focus is on teaching the child(ren) about the mathematical activities. However, the child's focus is on resolving a problem. In our data set, we had many situations that we could classify as belonging to this quadrant.

Apart from providing characterisation of situations with distinct didaktic makeup, another advantage of the didaktic space was that it allowed us to track changes in the focus of the teacher and/or the child(ren) within a specific situation. The dynamic nature of the interaction could then be described. In the future, this may allow us to determine whether the appearance of a specific mathematical activity or combination of activities might be related to the instrumental or pedagogical foci of the teacher and/or the children. Thus the didaktic space provides us with a way of conceptualising the "field of choices".

In the following sections, we re-analyse situations from our earlier work (Johansson et al., 2012) using the didaktic space model as an analytical tool.

Counting leaves

In an outdoor situation, the teacher had the children be pretend magpies and collect five leaves to place in hoops, which represented their "pantries". This example was chosen for reanalysis because it showed a common situation in which a child's focus appeared to be different from that of the teacher.

Björn:	Jag kan ränka, en, två, tre, fyra, fem	Björn:	I can count, one, two, three, four, five.
Lärare:	Fem, bra! Nu har ni fem stora löv i ert skafferi	Teacher:	Five, great! Now you have five large leaves in your pantry,

Originally, we classified this as an instrumental, Counting situation because the child initiated the counting, possibly to check if he had accomplished the task. We still classify it as instrumental for the child (IC). The teacher, however, had planned the situation so that the children would participate in Counting and thus learn something about the number 5. Thus, for the teacher it was a pedagogic situation (PT) even if she could not predict that the child would initiate the counting. In the didaktic space, it would be situated in quadrant ❹.

Walking along the bench

This second example illustrates how foci can change as a situation develops. Whilst playing outside, a toddler climbed on a bench and walked back and forth along it. The second picture in Figure 2 shows the child requesting assistance to get down, by raising her arms to the teacher. When the teacher did not pick the child up immediately, the child clambered down, after first gauging how far down she had to go.

Exploration of space is a feature of Locating. In this situation, the child seemed to have initiated her own learning about the spatial relations of being *up on* the bench *above* the ground, walking *along* the bench, *back* and *forth*, looking *down* to the ground. Hence, although there was no teacher actively involved, we originally considered that the purpose of the situation was pedagogic.

In the re-analysis, we pay more attention to the sequence of events. At first, the child did seem engaged in a pedagogic Locating situation (PC). As the teacher watched the child engage in Locating, the situation also seemed to be pedagogic for the teacher (PT). This part of the situation is located in quadrant ❸ (PC-PT).

Then the situation turned into an instrumental situation of Measuring for the child (IC) because she wanted to get down and now had a problem to solve. The child estimated the distance to the ground and compared it with her sense of her own size and climbing capability. First, she asked the teacher for assistance

Figure 1. Walking along the bench

by stretching out her arms. The teacher declined the child's request to be lifted down, probably because she wanted the child to engage in the problem of getting down by – physically and intellectually – combining her understandings of Locating and Measuring. Thus, it was a pedagogic situation for the teacher (PT). The situation now is located in quadrant ❹ (IC–PT).

The child then bent down and the teacher offered her assistance, perhaps because she decided the challenge of getting down was too much for the child. We interpret this as a change from pedagogic to instrumental for the teacher (IT). The child, however, declined the teacher's offer and climbed down without assistance so the situation can be said to be in quadrant ❶ (IC–IT).

Thus, during this situation, we see a move – in terms of the model – from ❸ (PC–PT) to ❹ (IC–PT) to ❶ (IC–IT).

Conclusion

This paper has explored the issue of how to respond appropriately to the research question "in what ways are preschool children engaged in which mathematical activities". As discussed earlier, other preschool mathematics education research which takes its understanding of mathematics from a school mathematics perspective can be considered problematic because of differences at the level of social practices. Our initial analysis of situations in a Swedish preschool (Johansson et al., 2012), using Bishop's (1988) 6 activities seemed to provide a more appropriate way to discuss the mathematics that children were participating in. However, it then became obvious that categorising situations from the researcher's perspective did not provide us with a sufficient detailed understanding of what was occurring in the video data. Although Walkerdine's (1988) distinction between pedagogic and instrumental purposes for situations seemed helpful in raising this issue, it then raised the issue of whether it was the children or teacher's focus in the situations that should be the basis for our analysis. The development of the didaktic space as an analytical tool has provided us with a more nuanced response to our initial research question. Nevertheless, these questions continued to remind us that our analytical choices influence what we can discuss when describing the mathematics of Swedish preschools.

It also seems likely that the didaktic space may solve some other methodology issues when researching the mathematical activities in which children engage. In Swedish preschools the curriculum is quite clear that children are not expected to reach any pre-set agenda of mathematical objectives. Instead, the objectives are about what the preschool should make available to children (Skolverket, 2010). As well, play has a central role in conceptions of how learning should occur. A research methodology such as the didaktic space is sensitive to and can capture the role of play. It also provides a way to interpret dynamic situations in Swedish preschools that can be useful in identifying the impact of the professional development initiatives now being provided to preschool teachers. This is because it provides a way of analysing data on what occurs in preschools both before and after an intervention of this kind, without relying on formal assessment of young children's mathematical knowledge.

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Preschool Teachers' Awareness of Mathematics

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Preschool teachers' expectations about what mathematics they should engage children in are generally centred about numbers and counting. However, the Swedish preschool curriculum and research into young children's development of mathematical understanding suggest that children can be offered a much richer set of ideas. In this paper, we examine material from a professional development course which indicates that discussing Bishop's six universal mathematical activities provided preschool teachers with a wider perspective for discussing the mathematics, which children in their preschools engaged with.

Introduction

Previous research on mathematics in early childhood education indicates a focus on counting and this may be related to the emphasis that it has in the school curriculum (Johansson, Lange, Meaney, Riesbeck, & Wernberg, 2012). However, the idea that mathematics in preschool is a watered-down version of school mathematics is problematic as it does not acknowledge young children's exploration of mathematical ideas in the same way that they explore other aspects of their world. Therefore, an alternative to seeing mathematics in preschool as a precursor to school mathematics, it can be considered closer to the experimenting and discovery type done by mathematicians (Devlin, 1999).

Not only do researchers seem to have a limited view on what mathematics in preschool could, but many preschool teachers share a similar or even more restricted view (Ginsburg, Lee, & Boyd, 2008). In Sweden, Björklund and Barendregt (submitted) asked 116 preschool teachers about what they focused on in mathematics in their preschools. They found that most teachers focused on numerical and spatial aspects.

> The absence of working with mathematical patterns is notable. Still, there is a tradition in Swedish preschools of working with beads, pearls or sorting games and play, but this may not be seen as a means for working with mathematical relationship and is thereby not problematized and scrutinized as a learning object and content within the goal-oriented education. One reasonable explanation for this could be that teachers have not reflected on the variety of

> aspects that mathematics consist of and thereby do not regard such activities as being part of mathematics. (Björklund and Barendregt submitted)

Even though the Swedish preschool curriculum, implemented in 1998 (Skolverket, 1998), included mathematical topics, such as measurement, shape, space and time, there is some uncertainty about the extent that Swedish preschool teachers introduce children to these ideas (Doverborg, 2006). This paper investigates preschool teachers' descriptions of mathematical activities that children engage with, after they have participated in a professional development course, and in the light of the revised Swedish curriculum. Using Bishop's (1988a; 1988b) six mathematical activities we analyse data provided by preschool teachers as a response to a prompt about the about the mathematics they present to children or consider that children engage in.

Bishop's six mathematical activities

Although not formally acknowledged, the mathematical objectives highlighted in the revised Swedish preschool curriculum can be traced back to Bishop's (1988a; 1988b) six mathematical activities (see Utbildningsdepartementet, 2010). In this background document is written:

> One way to concretely approach the objectives of the curriculum is to start from six historically and culturally founded activities. These activities may function as a structure in different context where mathematics can be discerned, explored and experienced. The activities provide opportunity to work with all objectives in mathematics in the preschool. They point out in which situations children and adults may need to use mathematics among other things. This entails that these activities not just connect to all objectives but also to the motives for the objectives. (Utbildningsdepartementet, 2010, p. 11; our translation)

Bishop (1988a) argued that the six activities were universal for any culture and labelled them as mathematics, with a small "m". The discipline of academic Mathematics, which he capitalised, included specific versions of the six activities. Bishop (1988b) summarised the six activities as:

> **Playing**. Devising, and engaging in, games and pastimes, with more or less formalised rules that all players must abide by.
>
> **Explaining**. Finding ways to account for the existence of phenomena, be they religious, animistic or scientific.
>
> **Measuring**. Quantifying qualities for the purposes of comparison and ordering, using objects or tokens as measuring devices with associated units or 'measure-words'.

> **Designing**. Creating a shape or design for an object or for any part of one's spatial environment. It may involve making the object, as a 'mental template', or symbolising it in some conventionalised way.
>
> **Counting**. The use of a systematic way to compare and order discrete phenomena. It may involve tallying, or using objects or string to record, or special number words or names.
>
> **Locating**. Exploring one's spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means. (p. 182)

In an analysis of video recorded data from one preschool in Sweden, we found that Bishop's six activities were all represented either through explicit interactions or incidentally through the provision of physical resources in preschools, (Johansson et al., 2012). However, this analysis was based on our interpretation of what we, as researchers, saw in the data. We were not sure that the teachers would have produced a similar analysis.

In this paper, we use Bishop's activities as an analytical tool for two reasons. One is the connection to the preschool curriculum (Utbildningsdepartementet, 2010). The second reason is that the objectives in the curriculum are not learning objectives for the children to reach and be assessed upon, but objectives for the preschools in regard to the learning opportunities they provide to children. Thus, we needed an analytic tool that ensured that we did not tacitly and inappropriately import school views on what counts as mathematics. By introducing teachers to the idea of Bishop's six activities and then asking them to describe what occurred in their own preschools from this perspective, we wanted to determine whether there was an even distribution of activities in the teachers' descriptions and to find out how the teachers reflected on using such a classification.

Collecting and analysing the data

The data consist of the final written assignment of 84 preschool teachers who had attended an in-service course focused on mathematics in preschool. Although not explicitly stated in the course syllabus, the course was based on Bishop's six mathematical activities. At the end of the course all participants were asked to answer the following three questions as a writing task.

- Vilka insikter har du gjort om dig själv, barnen och din praktik? (What have you learned about yourself, the children and your practice?)
- Vilka kunskaper har du utvecklat i och om matematik? (What knowledge have you developed in and about mathematics?)

– Beskriv hur du relaterar dessa kunskaper till hur barn lär och använder matematik. (Describe how you relate this knowledge to how children learn and use mathematics.)

Sometime after the course had finished, the teachers were asked if their assignment could be used as data. Of the 147 participants contacted, 84 responded favourably. When responding to these questions, teachers were expected to quote from the course literature and the preschool curriculum. Consequently, in the analysis of the data, statements about the curriculum or quotes from the literature were ignored. Instead we categorised the examples the teacher gave as examples of the mathematics, on which they were working or had begun to pay attention to, according to Bishops six mathematical activities.

Categorising the mathematical activities

Each teacher's response was read and examples were classified based on Bishop's (1988a, 1988b) descriptions of the six activities. When Macmillan (1998) used Bishop's six activities to classify preschool children's play, each example was labelled as only one kind of activity. However, in our data it was common that the teachers' examples could be classified as several activities simultaneously. Bishop (1988b) indicated that both kinds of categorisations were possible "the activities can either be performed in a mutually exclusive way or, perhaps more significantly, by interacting together, as in 'playing with numbers'" (p. 183). An example from our data is:

> Not to forget the winter which we are approaching, where one can build in snow and experience the concept of high and then on your own get to the top of the large snow pile and to experience it with your own body how difficult it actually is to climb that high

The part *"one can build in snow*" is categorised as Designing while "*experience it with your own body how difficult it actually is to climb that high*" is categorised as Locating. Thus, some examples could be in several categories while others were categorised as only belonging to one activity.

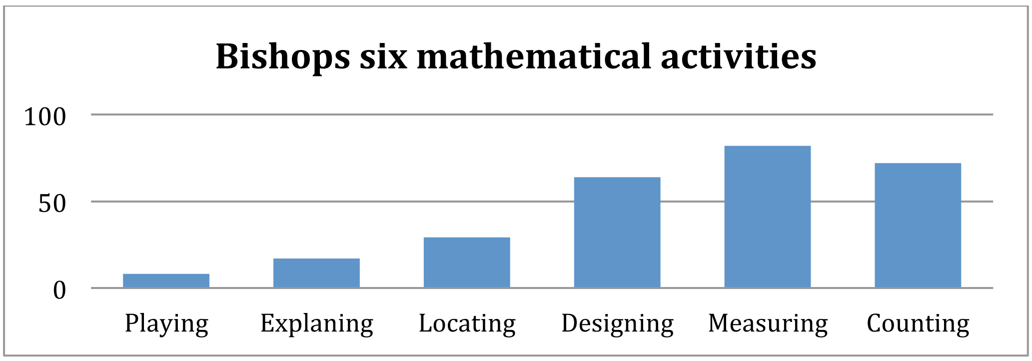

Figure 1: The distribution of preschool teacher's examples

Figure 1 provides an overview of how the examples were classified. The vertical scale is number of times each category appears. We can see that Designing, Measuring and Counting more often featured in the teachers' examples than the activities of Locating, Playing and Explaining. Playing and Explaining never occurred in isolation but always in relationship to one of the other four activities. In the next sections, we discuss why this can be and also give examples of each activity, while acknowledging that some examples could exemplify several activities.

Playing

Bishop's (1988b) mathematical activity Playing has similarities with the Swedish word "lek", but also some differences. According to Bishop, Playing consist of rules, which are more or less formalised. This has a connection to playing games, which in Swedish would be "spela", but could also be role-play, playing families and other kinds of play where the children imitate the real world in same way. This kind of play is "leka", but could include making decisions about the rules of the play (who is going to be the mother, father or dog, for example).

In the data, almost all of the preschool teachers used the Swedish word "lek" in connection to building play, movement play (bygglek, rörelseslek) etc. These were not counted as indicating the activity Playing, because it was not clear if the teachers were discerning the children's modelling, abstraction or hypothetical thinking which Bishop means is what makes Playing an mathematical activity. Rather it seemed that the teachers' conceptions of play were tightly connected to the curriculum which suggests that learning occurs through play.

> Children's play and creative activity cannot be separated from their learning because it is the same thought process which is activated when children express themselves in, for example drama play or drawing, as when children try to create understanding and solve a mathematical problem or inversely use mathematics or technology to make a stable construction in creative activity and building play. In that way mathematics becomes both a goal and a means [to achieve other goals]. (Utbildningsdepartementet, 2010, p. 5; our translation)

Play, as in "lek", is central in the curriculum and this seems to be reflected in preschool teachers' views on what should happen in the preschool. Therefore, it is not so surprising to see it mentioned but with a limited connection to the mathematical activity Playing. However, it is clear that if Playing is to be taken seriously as a mathematical activity and not just as a pedagogical practice, then future pre-service and in-service education needs to support teachers to gain a more comprehensive view of what Playing can and should be as a mathematical activity in preschools.

Explaining

According to Bishop (1988a), the mathematical activity Explaining answers the question “why”. Nevertheless, preschool children’s explanations often have a different form to those of adults or older children and so teachers may not always recognise them. In the following example, Sara describes a little boy playing and exploring with some sticks:

> One morning at preschool, I saw how little Emil from the toddlers group went around with a bunch of short sticks in one hand and a long stick in the other. ”What do you have there?” I asked. ”Many sticks,” he replied. ”What do you have in the other hand then?” “Not many!” said Emil. “Yes, that's right,” I said, “because you only have a stick.” He went to show his sticks to some of the older kids who were involved in building a hut from long branches. They had pushed one of the branches down in a snowdrift. Emil stabbed his long stick in the snow and looked alternately at it and the even longer branch, and said, “Mine is small!” At another time the same morning he sat on the ground and had lined up his sticks, two of them had the same length, which he had placed next to each other. He had the sticks in his hand throughout the morning before finally putting them in his pocket to go to lunch. In the afternoon he went out with the sticks in his hand! During the morning Emil explored a lot. He noted that the sticks were similar but at the same time different in shape and size. He distinguished and grouped parts into a whole, he categorised, formed pairs and more. He met adults who saw and put into words what he experienced and adults who had the ability to take his point. Teachers from the toddlers section had seen how important the sticks were for him. *Sara*

In this example, we can see from Sara’s description that Emil does clearly not use a verbal explanation but rather provides a form of explaining through categorising. Bishop (1988a) suggests categorising is one kind of explaining because it involves identifying a relevant attribute by which to make distinctions between items. Thus, there is an implicit explanation in deciding that an item belongs to one group rather than another. However, it would seem that the teacher identified this child’s actions as examples of the mathematical activities Measuring and Counting. All the examples which we categorised as the mathematical activity Explaining would perhaps not be recognised by the teachers as such but rather as other mathematical activities. Although one of the goals of the curriculum is that preschools should offer children opportunities to “develop their mathematical skill in putting forward and following reasoning” (Skolverket, 2011, p. 10), if the teachers do not recognise classification as a form of Explaining, they perhaps will miss opportunities to develop this activity.

Locating

Locating as a mathematical activity is about children locating themselves and other things in space. In the data, the examples included drawing, following maps and exploring the environment. Often position words were mentioned by the teachers. An example is the following:

> For example if the child should go on the slide, then I give the terms for what they are doing right then - now you climb up the ladder, then you should go down the slide. Another example - look the toy car went under the table, can you crawl under the table to retrieve it? *Marcus*

It was somewhat surprising that examples of Locating appeared relatively rarely in the examples that the teachers gave. Connections to space were mentioned in the 1998 version of the curriculum (Skolverket, 1998) as well as the revised curriculum (Skolverket, 2010). From our previous investigation (Johansson et al., 2012), we also had identified many examples that we could classify as Locating. It may be that exploring space and giving labels to children's experiences are so built into teacher's practices that they fail to recognise them as mathematical activities. However, it is clear that more research is needed to better understand why Locating, Explaining and Playing are not so well represented in teachers' examples.

Designing

Designing uses the image of a structure, often based on something in the environment to design an artefact. This design can be used to construct the artefact, but Bishop (1988a) is careful to point out that it is the mental actions of designing that makes Designing a mathematical activity. However, the focus of the preschool teachers seemed to be not so much on the designing of artefacts as of naming shapes and their particular features. The following is an example of this.

> When children do a puzzle, they must look at the shape, colour and image simultaneously. *Klara*

Being able to imagine the features needed in building is not mentioned, for example. Rather, the preschool teachers consider preschool children's choice of shapes in the construction of artefacts to be connected to the mathematical activities of Counting, as in the example below, or as Measuring.

> For example, at the lego table, the counting and calculating - I need a red narrow six door. *Fredrik*

This example was included as Counting because the teacher seemed to focus on the six. However, as the child seemed to focus on the features of the block needed for completing the building it was also classified as Designing. As the case with Locating, it seemed that the teachers did not recognise situations, in which the children engaged, that had links to other mathematical activities than Counting and Measuring. Clements and Sarama (2011) indicated that "geometry and spatial thinking are often ignored or minimized in … early education" (p. 133). However, we would suggest that it is not a case of ignoring or minimising the situations, but rather not recognising that they and the children were in engaged with Locating and Designing. The examples that teachers gave, which we categorised as Designing, fitted the more traditional view that preschool children should learn the names of two dimensional shapes. This raises questions about whether teachers need to be introduced more explicitly to Bishop's six activities in order for them to be able to recognise them in their own practice and to be able to provide opportunities for the children to engage in all of the mathematical activities put forward in the curriculum.

Measuring

There were more examples that were classified as Measuring than any other mathematical activity. Almost a third of these examples were about sorting or comparing in terms of size. Almost all of these were about length as was the case in the first example and in the example below.

> On the first occasion, they measured one child's length using pencils. Then they started making their own tapes which became too tedious after a while. Then they came up with the idea to take the bead jars to help them to measure the remaining children. This was not completed all the way when one of the children ran off to fetch blocks. One problem that arose for the children on the first occasion was that the kids realized that the boy was seven and a half pencils long were in fact the longest. Two of the other boys were equally long, but shorter than the boy who was measured using pencils. They were thirty jars and blocks long and the girl who became the shortest was twenty-eight. How could that be? *Lena*

The examples of the mathematical activity Measuring is not dominated by measuring with a specific tool but rather measuring or comparing with different kinds of objects. However, research on a six/seven year old child's out-of-school experiences (Meaney, 2011) suggest that there would be many other kinds of measuring than just length that children engage with. Consequently, it may be that the teachers need some more understanding of how to recognise potential situations in which to engage children in Measuring activities.

Counting

Counting was also a mathematical activity with many examples. The examples in this category include counting objects, sharing, determining how many remain after something is removed and pairing. The examples are from the everyday life in the preschool whereas the examples for the other activities were from playful or planned situations. The following example is typical in that sense:

> For example, at mealtime, setting the table, the children count how many children are going to eat, set the table the appropriate number of plates glasses and cutlery for the number of children. We share the fruit in halves quarters etc. *Agneta*

Ginsburg et al. (2008) and Björklund and Barendregt (submitted) suggested that preschool teachers' predominant view of mathematics revolves around numbers and shape names. It was therefore interesting to find that there were more Measuring examples than any other activity. It also seems that Ginsburg et al. (2008) concerns that US preschool teachers "generally do little to encourage counting or estimation, and seldom use proper mathematics terminology" (p. 6) were not relevant in regard to these Swedish preschool teachers who by introducing fractions involved children in a wider range of Counting activities. Nevertheless, it also seemed that some of the variety of activities that was documented was a result of the teachers attending the professional development. In the following quote, a teacher described how she had previously equated mathematics with the mathematical activity Counting.

> For example I have not used the word mathematics instead replacing it with "let's count".

Still there did remain some confusion over what mathematics could be developed from engaging in different situations:

> Finger Chants do not necessarily have a mathematical content, but it encourages mathematical thinking.

There seems to be a contradiction in this quote which suggests that some more research about what preschool teachers learn from engaging in professional development and how it affects their practice is needed.

Discussion and conclusion

This paper identified that although preschool teachers focused more on counting and measuring in their writing task, other mathematical activities were exemplified as well. Although some teachers recognised all the mathematical activities, it was apparent that most provided examples of Playing, Explaining and Locating but, by not explicitly labelling them as mathematical, may not have recognised them as such. It is interesting to note that sometimes teachers were aware that this was the case for the children, but not necessarily the case for themselves:

> The children 'talk' about mathematics without knowing it.

Nevertheless, although Bishop's (1988a) six activities were not explicitly described in the professional development, the teachers considered as beneficial having an alternative way of viewing mathematics was considered:

> Even to me, because I work with preschool, and work a lot with school, I somehow slipped into the school's working too much. Instead of transferring the preschool approach to school so it has become the opposite. Maybe because I have not had the right argument to advance preschool practices.

It is likely that the twelve years that teachers had engaged in school mathematics will have affected their perceptions of what and how they can engage children in mathematical situations in preschool. From our research, it seems that providing an alternative way of conceptualising mathematics may help preschool teachers take a broader view of what they should offer children. However, it also seems that changes will take time and a more explicit discussion of Bishop's six activities could be beneficial for future professional development programmes.

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Theorising the Design of Professional Development Web Modules

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This paper theorises the design of Skolverket's preschool and preschool class professional development web modules. By contrasting different models of teacher change, components are identified that designers of professional development materials may need to consider. Data from the decisions taken in designing the Skolverket project were analysed in relation to these components. From this analysis, it was found that some design considerations were not represented in the previous models. Consequently, a new model is proposed.

Introduction

As centralised education systems across the world try to raise the pedagogical content knowledge of mathematics teachers (Joubert & Sutherland, 2009), new professional development programmes often using information and communication technology (ICT) (Dede, Ketelhut, Whitehouse, Breit, & McCloskey, 2009) are being designed. Sweden is no exception to this, with the government initiating an extensive national professional development project (Skolverket, 2012). Teachers are expected to work in groups with web-based materials, known as modules (see Skolverket, 2012). Modules were divided into parts. For example, the preschool and preschool class professional development each had 12 parts. Each part contained four sections, A (individual studies), B (group discussion and planning), C (enactment/ observations in own teaching situations) and D (group discussion and follow-up). Several modules for teachers working at different levels of the school system have now been published by Skolverket, the Swedish Agency for Education (see Skolverket, 2012).

Each module is designed by teams from different universities at the bequest of Skolverket, who provide guidelines on the structure of the material as well as indications of the content to be covered (see Skolverket, 2012). As the designers of the professional development web-modules for teachers of preschool (concerning children 1 to 5 years old) and preschool class (children aged approximately 6 years old), we wanted to ensure that the material in the web modules would be in alignment with research on the professional development of teachers. In this paper, we describe previous models that theorised aspects of professional development and compare them with the decision-making process from our own design work. In particular, the model of Fishman, Marx, Best and Tal (2003) and their suggestions for the elements needed in the development of professional development material is examined.

Theorising the design of professional development materials

Although there are numerous models which theorise teacher change as a result of professional development (for example, Meaney, Trinick, & Fairhall, 2011; Clarke & Hollingsworth, 2002; Conway & Clark, 2003; Warren, 2008/2009), virtually no research-developed models about designing professional development exist. Similarly when Dede et al. (2009) set out a research agenda for online teacher professional development, they did not include a recommendation to theorise the design of material. Yet as Whitcomb, Borko and Liston (2009) stated:

> Attention to the preparation and support of professional development providers is essential to sustainability and scalability. The program must provide materials and resources that are sufficiently well specified to ensure that multiple facilitators in diverse settings can maintain integrity with the designers' intentions. Designers and early adopters must build the program's capacity by cultivating the knowledge base, experience, and leadership skills of novice professional development providers. (p. 211)

Without research about the design of professional development material, it seemed relevant to consider models of teacher change that occurred as a result of professional development. This is because professional development materials, through their implementation, are expected to contribute to teacher change. For example, Guskey's (1986) seminal model links professional development to enhanced student achievement (see Figure 1).

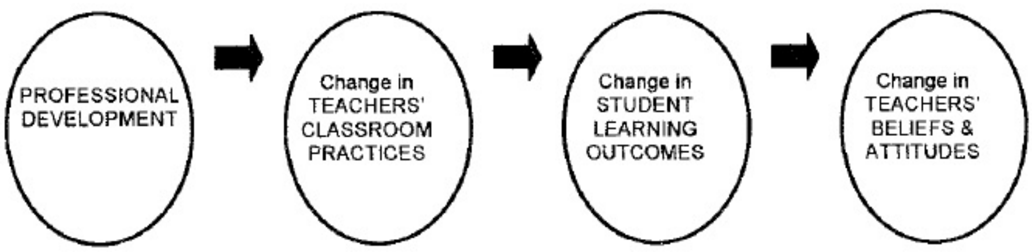

Figure 1: Guskey's model of teacher change (Guskey, 2002, p. 383)

Guskey (2002) considered that sustainable change in teacher practices only occurs after teachers' beliefs and attitudes had changed, but proposed that these changed as a result of seeing improvements in student learning outcomes that resulted from changes in teaching practices. Other models, such as Clarke and Hollingsworth's (2002), include the same components but do not consider the process to be linear. Rather they saw teacher change as being initiated as a result of changes in any of the other components.

Fishman, Marx, Best and Tal (2003) considered that teachers' beliefs, attitudes and knowledge changed as a result of professional development, which had an impact on enactment of classroom practices and awareness of student performance (see Figure 2). Compared to Guskey's (2002) model, enactment, in Fishman et al.'s model can be equated with "changes in teachers' classroom practices", student performance with "changes in student learning outcomes" and teachers' knowledge, beliefs, and attitudes with "changes in teachers' beliefs and attitudes".

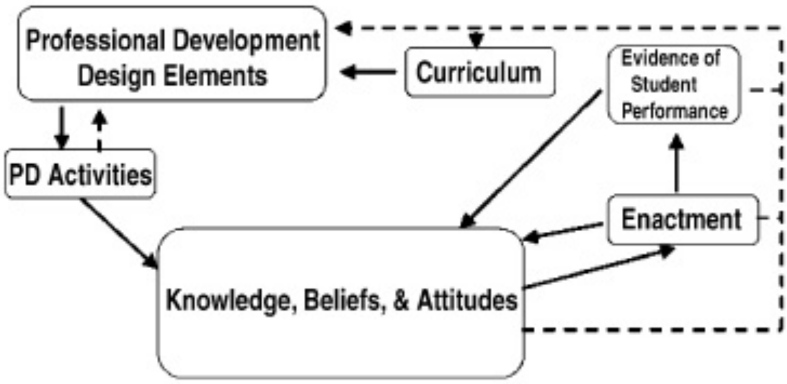

Figure 2: Model of teacher learning (Fishman et al., 2003, p. 645)

Fishman, et al.'s (2003) project is one of the very few that also considered the design of the professional development.

> There are four primary "elements" over which designers of professional development have control: the content of professional development, the strategies employed, the site for professional development, and the media used. These four elements can be combined in various ways to create professional development experiences for teachers. (Fishman et al., 2003, p. 646)

Content refers to the pedagogical content knowledge that teachers are expected to gain from participating in the professional development. The need for content learning is usually why teachers are considered to need professional development (Joubert & Sutherland, 2009). For Fishman et al. (2003), the curriculum was the starting point for considering the content to be covered. Still "participants in professional development can often come away with unintended learning that can include misconceptions or otherwise problematic understandings of the intended content" (Fishman et al., 2003, p. 647). Strategies are how teachers are expected to learn about the content. These can be considered as the professional development designers' teaching practices for supporting teachers' learning. Sites are the physical environments where teachers engage in the professional development. Fishman et al. (2003) do not take a position that one site is more beneficial than another. Rather they state that each site will have different affordances for the type of engagement expected. Thus, the choice of site(s) will have an impact on the strategies and media used. For Fishman et al. (2003), the media through which the professional development is conducted is the least important of the elements and is connected to both strategies and sites in affecting the format of the professional development.

We anticipated that to better understand our design process, it would be valuable to compare what we had done with Fishman et al.'s (2003) model, both the components that were related to Guskey's model (2002) and the design elements. By identifying if there were any components or elements that we had not considered, we would be able to improve our practices as professional development material designers.

Methodology

So that we could analyse our design process, we kept notes and audio-recorded the meetings that were held once a month from December 2012 until November 2013. Artefacts, such as contracts and email exchanges, were also kept. For this paper, we analysed a summary of our discussions from the first third (4 parts) of each module, which were developed simultaneously. The summary was used in the final preparation of these parts of the modules and acted as a reminder to ensure that the parts were in alignment with the agreements made during the first six months of work. The agreements came from our self-initiated discussions as well as reflections on a meeting with Skolverket's evaluation committee.

Table 1: Matrix of discussion points

	Content	Strategies	Site	Media
Teacher attitudes and beliefs				
Teacher knowledge				
Enactment				
Student outcomes				

Based on Fishman et al.'s (2003) model, we used a matrix with columns labelled with the 4 design elements and rows labelled with the components: teacher attitudes and beliefs; teacher knowledge; enactment; and student outcomes (Table 1). Although Fishman et al. linked the 4 design elements specifically to professional development tasks, we considered that tasks would be designed to affect each of Guskey's (2002) components. In line with Clarke and Hollingsworth's (2002) model, we separated knowledge from attitudes and beliefs as they seemed to require different kinds of design considerations.

The decisions in regard to the first third of the module were categorised as one or other of the four design elements by comparing each one to the Fishman et al.'s (2003) descriptions. Further, each decision was also categorised according to if it concerned attitudes and beliefs, knowledge, enactment (concerning something the teachers were asked to do in their normal preschool environment) or student outcomes (observation, assessment, documentation or discussion of own or other student's actions related to some activity or objective). In this way discussion items were slotted into the different cells of the matrix.

Examples and analysis

Our aim was to examine the general agreement between our design work and the components and elements of Fishman et al.'s (2003) model. As such, it was a qualitative study to see the level of agreement between what we had done and what seemed to be suggested as best practice. Therefore, we wanted to see if our decisions in the summary overview for the first four parts could be classified as fitting into all the cells in the matrix. Initially we were unsure that this would be the case. After the analysis showed that all cells could be completed, we were surprised to find that there were decisions which did not seem to fit any of the cells of the matrix. These are discussed in a later section.

Before discussing what was missing, we describe four examples of how the analysis was conducted. First we present an actual statement from the web material for preschool and then an explanation of the design team's intention with that statement which is connected to its classification in the matrix.

Example 1. Statement intended for Part 4D: *Update your pedagogical stance. Compare with what was written in 1A: What is same and what is different? Why? Compare with colleagues: What is the same and what is different? Why are there similarities and/or differences?*

The statement instructs the teachers to edit a text about their pedagogical stance that they wrote in part 1A. Writing and reflecting on an explicit pedagogical stance is a way of making one's beliefs and attitudes about teaching and learning visible. The instruction does not introduce new content, but asks teachers to compare changes in how they view their pedagogical stance from engaging with parts 1-4. As such it was a strategy about their beliefs and attitudes. Consequently, this was classified in the cell beliefs-attitudes/strategies.

Example 2. Document intended for Part 3C*: Observation matrix of forms of explanations.*

This observation matrix presents several ways of categorising children's explanations and is part of the content of the professional development. The intention was to let the teachers use this tool in their own practice, that is, themselves enact using the tool. Hence, the classification is enactment/content.

Example 3. Statement intended for Part 4B: *We have a range of documentation –*

How can we use this documentation? How can the documentation be shared or used with children?

This statement is concerned with student outcomes documented in a previous activity in the preschool environment. In an effort to deepen the discussion about the outcomes, teachers are to plan a subsequent learning situation in which the documentation is utilised by the children. In order to carry out this task, teachers must be aware of the interplay between the site in which the original documentation occurred and the site where the new situation will be enacted and how this might affect the new situation. The discussion is classified as student outcomes/sites.

Example 4. Video intended for Part 1A. *The video models how teachers could justify an observation's classification. Include the example of a child emptying a bucket.*

As a design team, we chose to build the modules around Bishop's 6 mathematical activities (Bishop, 1988). These are described in several texts in the module, but to connect the theory from Bishop to practice, the design team also wanted the teacher's to look at children engaging in different situations and see if they could identify the 6 activities. For this video was an important choice of media. We classified this decision as knowledge/media.

What was missing?

It was interesting to find that we could complete each of the cells in the matrix, more or less easily, but what was more interesting was that there were some points, which did not fit into any cell of the matrix. One important class of such discussions concerned relationships. For example, we had long discussions about how we addressed the users/readers and had decided that the plural form of you, "ni", would be used in instructions concerning activities and the singular "du" in instructions concerning reflections.

Concern about relationships turned out to figure in almost every discussion. In discussions about content and knowledge, attitudes and beliefs, we considered that it was important to build a relationship as designers of the materials with the teachers who were the users of the material, in a way that respected them as professionals. We also needed to consider how teacher tasks involved both providing a situation for children and documenting the children's interaction were affected by the relationships between the teacher and the children. It could be considered that Guskey's (2002) component of student outcomes as affecting teacher knowledge was a potential way of understanding the relationship between teachers and children. However, the actual examples of decisions that we were trying to categorise did not seem to fit easily into this row. Primarily, reflections on the task was done by prompting the teachers through discussion question. Thus, the decision to use discussion questions could be considered a strategy. Yet it seemed to require reflection about the relationship between teacher practices and children's participation and so was more than a strategy about student outcomes. It also seemed that implementing and documenting these situations would give teachers shared experiences that they could discuss with their colleagues. In this sense, such tasks also concerned and were affected by relationships between the teachers in the group and as designers, we had to take seriously the need for teachers to build relationships together.

Inter-relationships between Fishman et al.'s elements

In addition to the emerging category of relationships, some interesting relationships between Fishman et al's (2003) elements were apparent during the analysis. As exemplified by Example 3, most site considerations were a part of discussion involving strategies. Similarly, media choice also seemed to be closely connected to strategies. In cases where media choice was limited due to the web based nature of this PD, as designers we spent longer considering the strategies available us in designing tasks because of lack of choice about how a task could be presented. As well, when particular content only seemed possible to introduce through a particular media such as with the use of video in Example 4, media discussions also seemed to be strategic discussions. Consequently, sites, media and strategies seemed to be part of the same considerations rather than three separate considerations.

In contrast to Fishman et al.'s (2003) suggestion that media was the least important element, our circumstances meant we spent considerable amount of time discussing them. We wanted the teachers to watch videos, so that they could see typical Swedish preschool and preschool class children engaging in tasks from different mathematics education perspectives. Finding videos that were not exemplary teaching/learning but rather raised issues, took much time. Similarly, we wanted the teachers to document their and the children's participation and we considered that simply writing about it would not produce important reflections. Therefore, it seemed that the purpose of the tasks were related to media considerations and so it seemed unnecessary to split this decision-making between the component PD activities and the element media.

In the analysis of our discussions, many of them turned out to be related to the category of knowledge. As Skolverket's (2012) purpose was to "lift" teachers' knowledge about teaching and learning mathematics and consequently student performance, this is not surprising. However, research on the impact of teachers' attitudes and beliefs made us aware that we needed to provoke discussions about these and we chose to do this by asking questions for shared reflections. Similarly, enactment seemed related to strategies. Whereas enactment was concerned only with tasks done with children in their own preschool or preschool class, strategies seemed to be a larger construct because it enabled considerations of different kinds of tasks.

A model for designing professional development materials

The Fishman et al. (2003) model provided a good starting point for exploring our own work in designing the web-based mathematics education modules for preschool and preschool class teachers. However, there were difficulties in trying to operationalise it to understand our decision making process. The limitations that we found in existing models may be because their focus was on teacher change following the implementation of the materials, whereas our focus was on the types of considerations that professional development designers needed to respond to

Consequently, we propose a model specifically for the design of professional development material. It can be seen in Figure 3 and outlines the kinds of decisions that designers need to consider in developing materials which are likely to promote teacher change. Therefore although it draws on models of teacher change, it does so from the perspective of what is needed to design professional development material.

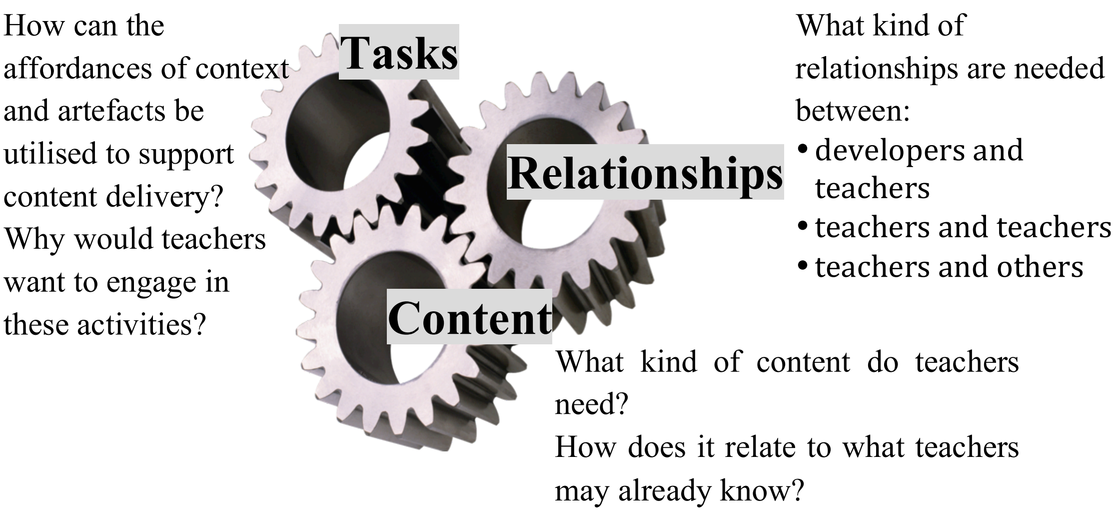

Figure 3: Professional Development Material Design Model

This model has three core components: the content; the tasks; and the relationships. These components interact with each other as decisions about one component is likely to affect the other two components, making it an integrated rather than linear model.

In projects, such as the one for Skolverket, content is the cog that drives the other two. This is because Skolverket identified the need for many preschool and preschool class teachers to improve their understanding of mathematics and how to develop mathematical task for young children to engage in. Even though designers often need to fulfil expectations of centralised education systems, there are likely to be some choices that designers can make in regard to content. In our case, we made the choice to present the content using Bishop’s (1988) 6 mathematical activities. We discuss our reasons for this in another paper, but here it is suffice to say that content decisions were related to the new knowledge that teachers were likely to need and how this related to the knowledge that they already had. This knowledge could be both discipline knowledge and/or pedagogical knowledge.

The second component in our model is to do with decisions about the tasks. This component is linked to Fishman et al.'s (2003) elements of site, media and strategies in relationship to the contexts and resources available for the teachers. The tasks connect to the content, but not just as a "deliverer", where teachers are asked to implement some aspect of the discussed content. Rather, the design of the tasks includes considering how they could be used to provoke teachers' reflections on their current practice and knowledge and relate these to new content perspectives. This means that the tasks were something that teachers enacted but also something that teachers needed to reflect on to gain other insights than were possible from merely reading about new content.

Our final component is relationships. We felt that it was a significant limitation in Fishman et al.'s model that there was no mention of relationships. For example, teachers are likely to gain more insights from their reflections if they are shared with other teachers. Thus, a relationship of trust between teachers is vital. Moreover, the content of the professional development might promote particular kinds of relationship with the children that teachers should reflect on which may result in changes to their existing practices. To contribute to the development of teacher-teacher and teacher-children relationships, it is important that the designers consider the relationships that they wish to be developed when designing the tasks. Tasks cannot only be designed to convey some content to individuals but they also need to develop appropriate relationships for maximising the potential for teacher change that will benefit children's engagement with mathematical activities. As well, we considered it important to consider the relationship between designers and users of the material. When designing, theories, ideas and experiences from research literature are packaged into professional development material for teachers who have experience, sometimes extensive, of what it means to support children's participation in mathematics activities in preschools. Consequently the material mediates between a scientific and a cultural (practice) perspective. Therefore, as designers we needed to have both an *expert* and a *philosopher perspective* (Certeau, 1984). The expert perspective concerns delivering certain, ideas, models, activities etc. The philosopher perspective uses the designer's scientific, specialist knowledge to ask questions, challenge routines and stimulate reflection.

In the new model, relationships, tasks and content are three equally important components in the design of professional development. As outlined earlier, decisions about one will affect the other two components. Although based on research into our own experiences, the usefulness of this model for designers will only be shown after extensive use and research in other projects.

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Structural and Pedagogical Diversity in Swedish Grade Six Algebra Classrooms

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This paper addresses the structural and pedagogical diversity in four Swedish grade six algebra classrooms. Drawing on video recorded observation and survey data from an international comparative video study, the results show wide variation of conditions for learning that highlight questions of inequality in decentralised educational systems such as that in Sweden.

Introduction

There is an on-going discussion in Sweden about inequality in schools based on a large variation in student achievement when measured in nationally administered standardised tests. The National Board of Education (Skolverket, 2012) reports that the variation in student achievement between schools has consistently increased since the late 1990s. Commonly this variation is attributed to socioeconomic and Swedish language skill issues. However, the variation in student achievement is great between classes as well as between schools, suggesting that the disparities could also be a result of pedagogical or structural variables. Since 1989 when much of the responsibility for administering public schools was decentralized and began to shift from the national to municipal level, there has been a series of reforms that have progressively strengthened local authority. There is, for example, no national regulation of the number of students in a class or any inspection of the textbooks used. In the regulations concerning school time tables, at the time of this study every student was entitled to 900 hours of mathematics instruction during their nine years of compulsory schooling. This equates to 100 hours of mathematics instruction per year or roughly 2.5 hours per school week (on July 1st 2013 the total was raised to 1020 hours)[1]. Adding to this, how these hours are distributed over the nine years of compulsory schooling is up to the local school to decide. A grounding principle descibed in policy documents is that there should be a wide variation of approaches and that although goals should be the same there are many ways to reach these goals. (Skolvernet, 2003). In short, local schools and districts have great structural and pedagogical freedom and are also financially regulated at the municipal level. In such a situation it is then perhaps not surprising that large structural and pedagogical variations have appeared over time that potentially have a significant influence on student achievement. In this paper the question examined is: given the decentralized nature of the Swedish school system, what diversity of structural and pedagogical conditions for learning exist in classrooms? We address this question in relation to the case of grade six mathematics classes.

The results reported in this paper are based on a subset of data from a comparative video-recorded study of mathematics classrooms in Sweden, Norway, Finland and the USA. A tentative comparison of some of the conditions for learning in the four countries indicates greater variation within than between countries (Partanen & Kilhamn, 2013). However, the Swedish data stands out as showing the greatest range of internal variation. This paper is an attempt to map the variability found in the Swedish data and raise questions that can be examined in subsequent studies. The results identify and unpack a diversity of conditions through detailed examination of weeklong sequences of lessons in four classrooms. While the data and results are focused on the situation in Sweden, they speak more broadly to the situation within decentralized educational systems.

Background

The project this paper reports on, VIDEOMAT[2] (see Kilhamn & Röj-Linberg, 2012 for a thorough description of the project), builds on previous studies of a similar character such as the TIMSS Video Study (Hiebert et al, 2003) and the Learners Perspective Study (Clarke, Kietel & Shimizu, 2006). It was designed as a comparative video study in mathematics education with a common focus on introduction of variables in algebra. As Clarke (2006) writes, an examination of classrooms across a variety of cultural settings and school systems makes our own educational assumptions visible and possible to challenge. The VIDEOMAT design as a cross-cultural video study seeks to view the practices of some algebra classrooms alongside the practices in others where the content area can be considered to be roughly the same. The overall aim of the VIDEOMAT project is not, as in the TIMSS Video Study, to identify and describe national differences in mathematics teaching, but instead to use the variation found in an international data set to compare classrooms to help reveal previously unidentified dimensions of algebra teaching.

Method

The research design for VIDEOMAT involved classes corresponding to Swedish grades six (last year of middle school, age 12) and seven (first year of secondary school, age 13) in each of the participating countries. This paper draws on a subset of the VIDEOMAT data including a sequence of video recordings of four consecutive teacher-planned lessons on introductory algebra from four Swedish grade six classrooms, teacher interviews, a written questionnaire completed by the teachers, and complementary material such as student work, lesson plans and curricular documents.

The results presented in this paper draw on video recordings and observational data from four Swedish grade six teachers, their pre- and post-interviews, and the questionnaire. The data collection was carried out during the 2011/2012 school year. During interviews and in the written questionnaire, the teachers were asked general questions about their teaching work in their grade six classes. As a reference point for student achievement in the classes, we use results from the nationally administered standardised test in mathematics[3] that the students took four to seven months after the observed lessons.

Participants

The four Swedish grade six teachers were recruited from three schools in the vicinity of Gothenburg[4]. All four described their decision to voluntarily join the project because they saw it as an opportunity for professional development. Two of the participating teachers were in the same school; school one teacher one (S1T1) and school one teacher two (S1T2). The other classrooms were in two separate schools; school two (S2) and school three (S3). The teachers in school one had three (S1T1) and 22 years (S1T2) of teaching experience, while both the others had 10 years experience. They were all educated as generalist teachers (Swedish: klasslärare) but due to frequent reforms in Swedish teacher training programs, their educational backgrounds were all slightly different. They all worked in schools with a traditional middle school structure where generally one teacher is expected to teach the same group of students in most subjects from grade four through grade six. The three schools represented different demographic regions. School one is situated in a small rural municipality close to Gothenburg, school two is located in the Gothenburg archipelago, and school three was an inner-city school. All three schools were public schools and none of them were located in extremely high- or low-income areas. Although there were students in all schools who did not have Swedish as their first language, all the students could comprehend and speak Swedish well, Swedish was the language of instruction and the common language of communication among students.

Analysis

As a theoretical frame, the VIDEOMAT project as a whole is placed in the field of sociocultural research and therefore focuses on the activities, artefacts and types of interaction that took place in the classrooms. As a first step for organizing the video-recordings collected from the four classrooms, a coverage code system was created describing the content in the videos and partitioning them into smaller more manageable instances of activity. This coding scheme drew on the codes used in the TIMSS Video Study (Hiebert et al, 2003) with adaptions made to reflect the particular activities found in the classes in our data. To meet the interest in the introduction of variables in algebra in the VIDEOMAT project, codes were attuned to identify the introduction of new content and the use of variables in written work. Following the approach taken in

TIMSS, the coverage codes we used are mutually exclusive descriptions of what can be identified as the main activity for a particular instance of class time. When the teacher orchestrated a shift of activity or a majority of students shifted into a new type of activity, a new code was applied. For the purpose of consistency between coders, we took the shortest time for a coded instance to be one minute.

The coverage codes are descriptive of the type of activity in a classroom (e.g. *No Mathematics, Mathematics Whole Class* activity or *Mathematics Student Work*). Whole class activities are coded as either *Introduction* or *Follow up* to distinguish instances where the teacher gives instructions or introduces new content from instances where s/he reviews student work or revisits previously introduced content. Beyond the scope of the data presented in this paper, whole-class activities have been further coded to identify different types of interactions between teacher and students. All mathematical activity that was not whole class activity was coded as *Student Work*, either *Individual* or *Group*, where group indicates that the students worked on the same task together in pairs or small groups. In such group work activities, documentation and written work were coded as being conducted *Individually*, as a *Group* or not at all (*None*). Student individual work was most often identified in instances when students worked individually from their textbooks or with worksheets, with the teacher walking around interacting with individuals. In these situations student-to-student interactions occasionally occurred but not for the majority of students and not with a consistent focus on shared mathematics tasks. Figure 1 shows the coding system at the level of analysis reported on in this paper.

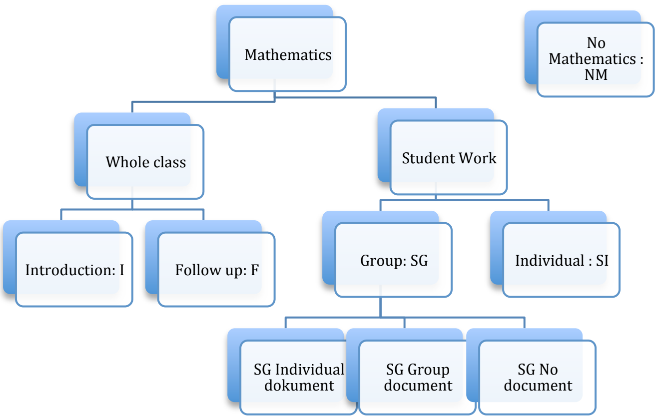

Figure 1: Coverage coding referred to in this paper.

Results

Drawing on the coding of the types of activity undertaken in the four Swedish classrooms along with interview and questionnaire data, our results shed light on the diversity of structural and pedagogical conditions present in Swedish mathematics classrooms. The results are divided between the interrelated issues of structural variability, reflecting such considerations as class size, homework policy and teaching responsibility, and pedagogical variability, addressing such factors as types of activity and proportions of class time used.

Structural variability

The most obvious variation in structural conditions amongst the four classrooms was the number of students present. The smallest class (S1T2) had only 13 students while the largest class (S2) had 30. S1T1 had 18 students while in S3 the situation was complicated by a schedule of whole and half class lessons. Here the organisation of lessons meant that, while the class had 20 students, every second lesson was a half class lesson with only 10 students. Along with class size, there was also significant variability in the amount of time dedicated to mathematics in the four classes. In the survey we asked how much time per week students were scheduled for mathematics (A) and how much time per week the teacher estimated that s/he spent on preparation and correction of student work in mathematics (B), see table 1.

Table 1: Structural variables concerning time

	S1T1	S1T2	S2	S3
A: mathematics per week (min)	160	160	180	200
B: teacher preparation per week (hours)	1-2	1-2	6-10	3-5

A clear variation in time for both teaching and preparation is seen among the schools but not between the two teachers in the same school. The time allotted to mathematics instruction highlights an inconsistency in the application of rules from the national board of education, while the difference in teacher preparation time combined with the survey and interview data suggests inconsistency in how teachers are expected to distribute their preparation time.

Another structural variable with similar diversity across schools concerns school level policies and actual practices related to homework. On the questionnaire three items addressed homework. One shows the number of assignments students receive per week (C), and the length of time students are expected to spend on them (D), see table 2.

Table 2: Homework assignments

	S1T1	S1T2	S2	S3
C: assignments per week (avg)	<1	<1	1	1
D: time on each assignment (min)	<30	<30	30-60	30-60

Consistent with the situation in many Swedish schools (Forsberg, 2007), homework was scarce in the participating classes in our study. However, despite the overall limited amount there was a distinct variation among schools with one homework assignment per week forming an important part of the instructional practices in two of the schools (S2 and S3) but not in school one.

The other survey item connected to homework concerned extra curricular mathematics. The question asked was: are there any situations outside of the ordinary mathematics lessons when you know or believe your students spend time learning mathematics? In school one, where very little homework was assigned, the teachers described no extra curricular mathematical activities. However, in the two schools where homework was assigned consistently, the school offered homework assistance once a week and both teachers noted that around six students in their classes regularly attended.

A fourth structural variable relates to teacher responsibility and presence during mathematic lessons. Although the generally recognized model for Swedish grade six classes is one teacher per class, this was not the actual situation in three of the four classes. In S1T2 there was one teacher present during our observation, however another teacher had the overall responsibility for mathematics teaching in the class and the observed teacher only taught some mathematics lessons. In S2, two teachers also shared the class; one teacher had the responsibility for mathematics instruction but a second teacher sometimes assisted. Similarly, in S3 one teacher was responsible but there was sometimes a special needs teacher or a teacher assistant present. The diversity visible in our four classes shows that the uniform model of one teacher per class at the middle school level may not represent the practice in Swedish middle schools. This reflects the wide variety of structural conditions we found in the schools.

Pedagogical variability

While there was significant structural variability between schools, we also identified a number of pedagogical variables that show diversity between classrooms even within the same school. The four pie charts in figure 2 show the coverage coding for the four grade six classrooms in terms of the percentage of lesson time spent on various types of activity. As addressed earlier in relation to survey question (A), the total amount of lesson time per week varied among schools. Since the length of each lesson also varied and in some schools often deviated from the set timetable, the four coded algebra lessons each had different lengths of lesson time. To address this, we observed and coded the four consecutive lessons from the point of view of student experiences of mathematics lessons rather than scheduled class time (e.g. in S3 an 80 minute long half-class double lesson was repeated in each group but only counted once). The total coded lesson time across classes was as follows: S1T1 - 2 h 27 min; S1T2 - 2 h 42 min; S2 - 3 h 57 min; S3 - 3 h 6 min.

We can see from the pie charts in Figure 2 that the variation is large for several types of activity. Between three and 76 per cent of the available classroom time was spent on individual student work, and between zero and 36 per cent was spent with students working in groups. The distinct variation in amount of time spent on non-mathematical activities can partly be connected to different types of student work. In S3, where most of the time was spent on Student Individual work, only three per cent of the lesson time was used on organization and classroom management. In S2, where students worked frequently in groups, time was spent moving students around, reorganizing the classroom and discussing rules for group work. All teachers spent time introducing new concepts (between 14 and 31 percent of lesson time), but the variation was greater in relation to the amount of time spent on whole class follow-up activities (between 5 and 36 percent).

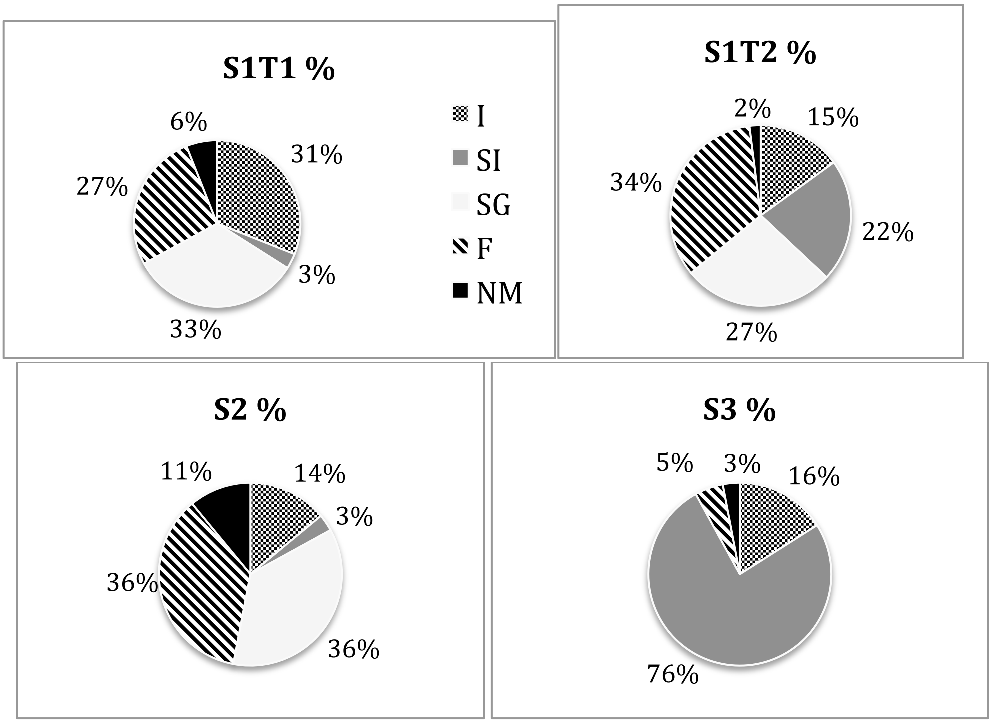

Figure 2: Pie charts showing the distribution of lesson time by type of activity expressed in percentage of total lesson time. Codes: Whole class Introduction (I), Whole class Follow-up (F), Student Individual work (SI) Student Group work (SG), Non Mathematical activity (NM)

Another feature of the classroom activities captured how much students' practiced expressing mathematics in written documents. Figure 3 shows the distribution of time spent in each class on student work, differentiating between different types of writing practices in individual and group work.

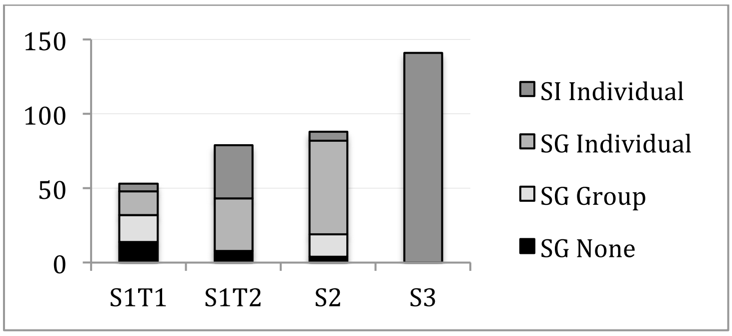

Figure 3: Bars showing lesson time in minutes spent on student work differentiating between different types of documentation.

Student Individual work (SI) assumes individual documentation, and Student Group work (SG) was either documented Individually, in a shared Group document or Not at all. As is clearly visible in Figure 3, the practices of writing in the algebra classrooms were all distinctly different.

A third pedagogical variable we identified as showing large variation was the use of textbooks and teaching materials. In the planning interview all four teachers stated that they use the same textbook, but as it turns out they use it very differently. S1T1 and S1T2 had recently invested in an activity box containing teaching materials with hands-on algebra and patterning activities[5]. In part due to this recent purchase, both teachers decided not to use the textbook at all for the unit on algebra. Instead they used activities and material from the box and additional worksheets from the National Centre for Mathematics Education[6]. In S2, tasks were taken from the textbook and worksheets, and were often projected onto an interactive whiteboard. However, the students in the class only had access to paper and pencil and did not use their own copies of the textbook during the time we observed. By contrast, in S3 the textbook was used in a traditional manner where all students had their own copy and worked through the sequences of tasks at their own pace in the order provided by the authors. This variability in use of the same textbook highlights the potential differences in pedagogical conditions even given the same or similar structural conditions. Combined, the spectrum of structural and pedagogical conditions identified give rise to different classrooms with variable opportunities for learning.

Discussion

The results presented are based on a video study involving only four classrooms. We do not know to what extent these classrooms represent Swedish grade six mathematics classrooms at large. However, the diversity of structural and pedagogical conditions found warrants questions of how valid national characterisations of schooling can be, particularly in relation to largely locally controlled systems such as that in Sweden. Many of the differences identified largely depend on decisions made at school and school district levels. The large differences between how much mathematics education a sixth grade student is offered (160 or 200 minutes per week), to what extent homework is used as a complement to school instruction (less than once a week or up to 60 minutes per week, with or without homework assistance at school), and how much preparation time teachers spend preparing for their mathematics lessons (between 1-2 and 6-10 hours per week) indicates an inequality in the conditions for learning that students are offered. The various different ways of organising lessons and classes, with class sizes of between 13 and 30 students, half-class lessons, assistant teachers and shared responsibility for mathematics instruction may be a result of pedagogical considerations, but they may equally be a result of financial considerations. In addition, there is a pedagogical diversity in how lesson time is spent and how textbooks and other teaching materials are used that our interview and survey results suggest are largely a consequence of decisions made by individual teachers or teacher teams while clearly being connected to structural and pedagogical conditions decided upon at other levels.

The presented results have illustrated structural and pedagogical diversity found in four classrooms in three different schools in the Gothenburg vicinity. Diversity was also great when student achievement was measured in these classrooms. The results on the grade six national test in mathematics in the three schools in spring 2012 showed a variation between 51,1% in school one and 96.4% in school two for students who demonstrated reaching a level of learning expectations considered to meet national standards for their grade level. While it is important to recognize that this study offers no evidence of a correlation, it does raise questions about possible relationships between structural and pedagogical variables and student achievement. For example received wisdom often assumes that large class size is negative for student achievement while in this study the students in the largest class ranked the highest on the national test.

There is much research about possible factors that may influence a teacher's pedagogical decisions, such as their mathematical content knowledge or beliefs (e.g. Boaler, 1999; Hall et al, 2008). Different pedagogical approaches, such as making use of written work, interaction, and whole class feedback and follow-up identified in this study may be a result of a knowledgeable teacher's adjustment to the different needs of his or her students. However, such decisions may also be a result of differences in a teacher's knowledge or beliefs. For future research we suggest that the variation we have seen in these four classes is investigated on a larger scale to see if the diversity is as great in Sweden as a whole as it was in our sample. Our results raise questions about the connection between pedagogical diversity and student achievement and indicate that structural and pedagogical variables should be seriously considered alongside such factors as socioeconomics and language skills. If a future aim is to slow the increase in inequality amongst Swedish schools, we may have to reconsider the grounds on which achievement levels are explained and on which decisions about students' mathematics education are made.

Notes

1. http://www.skolverket.se/laroplaner-amnen-och-kurser/grundskoleutbildning/grundskola/timplan
2. Financed through a grant from The Joint Committee for Nordic Research Councils for the Humanities and the Social Sciences; NOS-HS (Project No.: 210321/F10).
3. http://siris.skolverket.se (retrieved 2013-10-21)
4. Information was sent out with the help of school board mathematics specialists
5. NTA(Naturvetenskap och Teknik för Alla)-lådan: Mönster och Algebra [*NTA(Science and Technology for All)-box: Patterns and Algebra*]
6. Activities from the National Centre for Mathematics web page e.g. "Strävorna"

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Incorporating the Practice of Arguing in Stein et al.'s Model for Helping Teachers Plan and Conduct Productive Whole-Class Discussions

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How can pedagogical models support in-service and pre-service teachers in the complexity of orchestrating productive mathematical whole-class discussions? The overarching aim of this paper is to elaborate on a newly developed model to make it an even more useful tool for teachers to manage the challenging task of conducting productive whole-class discussions. Analyses of audio-recorded interviews and video-recorded whole-class discussions with a proficient mathematics teacher result in principles for how student solutions can be sequenced in order to take into account argumentation as well as connection-making in whole-class discussions. The findings suggest broadening the last practice in the five practices model to also incorporate the practice of arguing.

Introduction

Mathematical discussions that focus on important relationships between mathematical ideas in students' different solutions to demanding problems can be seen as a significant ingredient in high-quality or ambitious mathematics teaching (Cobb & Jackson, 2011; Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010) that aims at developing students' mathematical competencies (NCTM, 2000; NRC, 2001). For teachers to learn the challenging task (Brodie, 2010) of orchestrating such productive whole-class discussions that take both students' participation and important mathematical content into consideration (cf. Ryve, Larsson and Nilsson, 2011), there is need for supportive routines of practice (Franke, Kazemi, & Battey, 2007) or instructional practices (Cobb & Jackson, 2011). Stein, Engle, Smith and Hughes' (2008) model of the five practices anticipating, monitoring, selecting, sequencing and connecting aims at helping teachers plan the orchestration of productive whole-class discussions that both build on student ideas and highlight and advance important mathematical ideas and relationships. Stein et al.'s (2008) model is designed to be used in in-service and pre-service teacher education as a tool for mathematics teachers at all school levels to learn to conduct productive mathematical discussions that focus on connections between different student ideas and between student ideas and key ideas. However, both arguing and connecting constitute the keys for creating opportunities in discussions for extending student thinking (Cengiz, Kline, & Grant, 2011). The overarching aim of this paper is to further elaborate on Stein et al.'s (2008) five practices model in order for teachers to manage to conduct productive whole-class discussions that focus on argumentation as well as connection-making. Stein et al. (2008) emphasize that much more research is needed on how to sequence student solutions and a particular aim of this paper is to contribute to that area of research.

Conceptual framework

Stein et al.'s (2008) five practices model for helping teachers plan the orchestration of productive mathematical discussions is central in my analysis and the model itself is also analyzed. The five practices in Stein et al.'s (2008) model are: anticipating student responses to cognitively demanding tasks, monitoring student responses during the explore phase, selecting student responses for whole-class discussion, purposefully sequencing student responses and connecting different student responses to each other and to key mathematical ideas. Each practice builds on and benefits from the practices that precede it. The five practices have clear connections to teaching practices in Japan, where teachers often organize a complete lesson around students' various solutions to a single problem in a whole class setting (Shimizu, 1999). Crucial Japanese instuctional practices include anticipating student approaches and observing or monitoring students' work, looking for good ideas "with the intention of calling on those students – in a certain order – in the subsequent discussion" (Shimizu, 1999, p. 109). The order is critical for making connections among student ideas.

The basic assumptions underlying the five practices model as articulated by Smith and Stein (2011) are that we learn through using others as resources in social interaction, sharing our ideas and participating in co-construction of knowledge (cf. Cobb, 2000; Cobb, Stephan, McClain, & Gravemeijer, 2001). To support student learning, Smith and Stein (2011) accentuate the importance of encouraging students to evaluate their own and other students' mathematical ideas. However, Stein et al.'s model provides no explicit support for teachers regarding this aspect. I will operationalize this aspect in my elaboration of their model to take into account argumentation as well as connection-making.

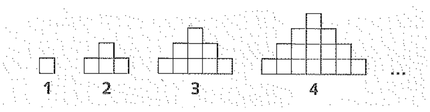
Methodology

The primary data source for this paper comes from a project which I conducted in collaboration with a very experienced and proficient teacher regarding problem solving discussions. I observed the teacher during eight days in one school year without making interventions. I had a particular focus on the teacher's orchestration of whole-class discussions based on students' different solutions to challenging mathematical problems. Data consists of video-recorded lessons focusing on the teacher during whole-class discussions, audio-recorded teacher interviews before and after every lesson, audio-recorded student interviews, audio-recorded teacher meetings and collected student solutions. Stein et al.'s (2008) model serves as the primary framework for analyzing the data. Data from this project will feed into my ongoing work on suggesting elaborations of Stein et al.'s (2008) framework, together with data from several intervention projects that I have conducted (Larsson & Ryve, 2011; 2012). In these intervention projects I collaborated with teachers learning to conduct whole-class discussions of students' different ideas. One project involved all mathematics teachers in grade 6-9 at one school during the course of two years.

Analysis and results

As an illustration of how the proficient teacher reasons when she sequences student solutions, I will now go into a whole-class discussion in 6th grade of students' different solutions to the problem Winners' stands. I will relate this particular discussion to principles for sequencing student solutions to take into account argumentative aspects as well as connection-making aspects.

Winners' stands



How large perimeter and area has winners' stand number:

a) 15 b) 20 c) n

In Table 1, you find the student solutions for area in the sequential order that they were brought up in whole-class discussion. In fact, the solutions correspond to Mason's (1996) three major approaches for how algebraic formulas are constructed: (1) finding a recursive rule of how to construct the next term from the preceding terms (Edward and Anna), (2) manipulating the figure to make counting easier (Anders and Pia, Fredrika and Carl), and (3) finding a pattern which leads to a direct formula (majority of the students).

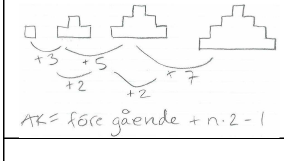
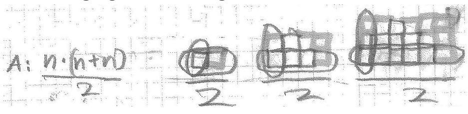
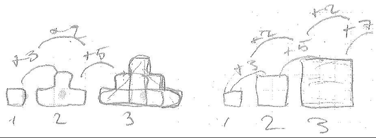
Edward's and Anna's solution Preceding figure + bottom row 	Anders' and Pia's solution Rearranging into rectangles 
Majority of the students' solution Seeing number pattern in a table a) 15·15=225 cm² b) 20·20=400 cm² c) A=n·n	Fredrika's and Carl's solution Rearranging into squares 

Table 1: Student solutions for area of winners' stands in sequential order.

In the following excerpt, we enter the discussion from the start when Edward explains his and Anna's formula for the area of the winners' stands (see Table 1).

1	Teacher:	Let's start with area. This is one solution. Eeh then we have (.) let's look at winners' stand number 1, number 2, number 3, number 4 [points at the figures one at a time] and shown that the difference is 3, 5, 7 [points at the differences one at a time] and that it then increases with 2 and 2 [points at the twos one at a time]. And then your formula is (.) could you just explain your formula.
2	Edward:	So the number of squares equals the preceding figure before, because it's them you can- you can see that they sit and then you have just added a bottom. And the bottom equals the number of the figure times 2 minus 1.
3	Teacher:	The number of the figure times 2 minus 1. So for example in figure number 2, no number 3 it is 1, 2, 3, 4, 5
4	Edward:	And 3 times 2 equals 6, minus 1 is 5.
5	Teacher:	Does anybody understand what kind of formula they've written here?
6	Students:	Yes.
7	Teacher:	You do understand?
8	Fredrika:	Yes.
9	Teacher:	Fredrika, could you explain the formula to see if we underst- if we all understand.
10	Fredrika:	So look, it's like this. Eeh, if-
11	Teacher:	Edward, listen to see if, if Fredrika understands what you mean.
12	Fredrika:	If we deal with, if we say that we're on figure number 3
13	Teacher:	There [points at figure number 3]
14	Fredrika:	Yes. Eeh (.) okay you (.) if you look at the preceding number 2, before [teacher points at figure number 2] it looks like that. And the difference between that and number 3, it's that you have added a bottom in it, a new floor farthest beneath. If you see that. Yes.
15	Teacher:	Mm.
16	Fredrika:	So then it's the preceding figure
17	Teacher:	The one up here [points]

18	Fredrika:	Yes, exactly. And then plus this bottom which is (.) so n so the figure times 2 minus 1.
19	Teacher:	So 3 times 2 is 6, 6 minus 1 is 5. 1, 2, 3, 4, 5. Now I understand. Does anybody else than I understand?
20	Students:	Yes. Mm. I understand.
21	Teacher:	Sanna, do you understand?
22	Sanna:	Yes.
23	Teacher:	Hannes understands?
24	Hannes:	Yes, but I don't get how- how would you find out the preceding figure?
25	Student:	No.
26	Edward:	I know, that's our little problem, that if you don't know that then you can't really use this one.

To begin the discussion with Edward's and Anna's recursive formula in which the area for one winners' stand builds on the area for the preceding winners' stand serves as a springboard for the rest of the discussion since the limitations with the solution are made explicit by Edward himself in [26] after Hannes' question in [24]. The teacher chose to begin with Edward's and Anna's solution because "there was still a problem to solve" (interview after discussion). The teacher does not authoritatively evaluate Edward's and Anna's solution, but instead facilitates for the students to evaluate each other's solutions which is salient for a dialogic approach that takes different points of views into account (Ruthven, Hofmann, & Mercer, 2011). The teacher first lets Edward explain his and Anna's solution ([2] and [4]). Then the teacher repeatedly asks if anybody understands ([5], [19]), after which she follows up with asking if specific students understand ([7], [21] [23]) and asking Fredrika to actually explain how she understands Edward's solution ([9]), emphasizing the importance that Edward listens carefully to see if Fredrika understands what he means ([11]). When the teacher asks if Hannes understands he raises the question of how you can find out the area for the preceding figure ([24]), which is a clear limitation to the solution that Edward already seems aware of ([26]). The teacher confirms in the interview after the discussion that Edward was in fact aware of this limitation before the discussion but that "Edward was completely convinced that, when he presented, that certainly all of them had that problem" and "that was why he was so sure and could explain that yes, if I only knew what the preceding is".

After this exchange, Anders' and Pia's solution of rearranging the winners' stands into rectangles (see Table 1) is discussed. According to the teacher "they realized later that there was an easier method, but they were exceedingly happy when they drew their rectangle". Their solution is evaluated by the students to be a smart solution that resembles a solution to another problem that they have previously worked with, but that there exist easier solutions to this problem. A majority of the students have seen from the number pattern in a table that the formula for the area of the winners' stands can be expressed as n·n. This is the next solution to be discussed very shortly (see Table 1) and Anders states that he regards it as much easier than his own solution. Finally, the teacher highlights Fredrika's and Carl's rearrangement of the winners' stands into squares to find out the formula n·n (see Table 1).

If we step back from these solutions for a moment, we can see how the recursive solution serves as a springboard for argumentation. We can also imagine how an early introduction of the solution from the majority of the students could have affected the quality of the argumentative aspects of the whole-class discussion. If a majority of the students have already received confirmation in the beginning of the discussion that their own solution is correct, there is a considerable risk that they do not listen as carefully to the other student contributions and that they do not contribute by putting forward arguments for or against the validity of different solutions. I will now go further into how the first four practices, in particular sequencing, are critical for argumentation as well as for connection-making.

Anticipating, monitoring, selecting and sequencing to promote argumentation as well as connection-making

Clearly, the first four practices in Stein et al.'s model are crucial in order to create opportunities to connect student solutions to each other and to key mathematical ideas (cf. breadth and depth connections in Ma, 1999). However, anticipating, monitoring, selecting and sequencing students' solutions are also crucial for argumentation during the whole-class discussion. When anticipating student solutions, in particular misconceptions, an important aspect for the teacher is to prepare for the kind of arguments that students are likely to present during whole-class discussion. When monitoring student ideas, my findings suggest that it is critical that the teacher does not disclose to the students whether their solution is correct or not. The reasons are both related to the problem-solving process and to the quality of the argumentation during the subsequent whole-class discussion. The proficient teacher states that "It's quite hard but it's extremely important that you don't tell if it's right or wrong because then you have removed what's the problem in the problem" (interview, Oct 27, 2011). This important aspect of the monitoring practice needs to be emphasized in Stein et al.'s model. If the students ask if their answers are correct during the problem-solving process, the proficient teacher asks questions to activate the students as owners of their own learning (e.g. "What do you think, is it right or wrong?") or as instructional resources for one another (e.g. "I don't know, discuss it with your friend."). (cf. Wiliam, 2007).

When selecting and sequencing student solutions, Stein et al. (2008) suggest that you start with either: a strategy based on a common misconception, a strategy that is particularly easy to understand or a strategy that a majority of the students have used. The first two suggestions are in line with my findings. Starting the discussion with a strategy based on a common misconception give the students the opportunity to straighten out their misconceptions before going deeper into the discussion of different correct strategies. Starting the discussion with a strategy that is particularly easy to understand resonances with the goal of accessibility (Stein et al., 2008) so that as many students as possible are able to follow and contribute to the discussion.

However, my findings suggest that there are some problems with the third suggestion. Starting the whole-class discussion with a solution that a majority of the students recognize as their own, or very close to their own, may compromise argumentation during the discussion. Instead of starting with a solution that many of the students have made, the proficient teacher places a common type of solution among the last ones in the sequence (see Table 1), or even skips it totally if it is very well-represented in the class. Analysis of whole-class discussions and interviews with the proficient teacher result in the following principles for sequencing student solutions:

1. an incorrect solution that seems reasonable that gives rise to argumentation (cf. common misconception in Stein et al., 2008)
2. a correct solution that is well structured with each step written where you can easily follow the whole line of thought (cf. goal of accessibility in Stein et al., 2008)
3. different solutions that show variety among solution strategies and representations with the potential to generalize to key mathematical ideas carefully considered, sequenced as more and more difficult to understand
4. (a solution that a majority of the students have made)
5. an elegant solution that makes the problem appear easy

The suggestion that teachers should not only discuss the students' correct solutions but also their incorrect solutions builds on the view that errors and misconceptions are "a normal part of coming to a correct conception" (Brodie, 2010, p. 14). The importance of giving students the opportunity to correct their own mistakes in front of the class is emphasized by the proficient teacher in my study, in line with Boaler and Humphreys (2005). The teacher states that "they get a chance to say to the whole class: Ah, I made a mistake here, but I should have done like this instead". A 7th grade student in her class expresses herself like this: "While you explain, some understand that they have made a mistake, so they learn while they explain".

My findings indicate that the first four practices are crucial not only for the practice of connecting but also for the practice of arguing which needs to be properly addressed within Stein et al.'s model. Therefore I suggest broadening the last practice in Stein et al.'s model to incorporate the practice of arguing.

The last practice in the model: Extending by arguing and connecting

Extending student thinking has to do with further development and challenging of student thinking (Cengiz et al., 2011). Arguing and connecting actually constitute the main part of creating possibilities in discussions for extending student thinking. Cengiz et al. (2011) state that “recognizing moments for building new connections or addressing misconceptions seems to be key in creating opportunities for extending student thinking” (p. 362). Misconceptions can be addressed by challenging them with mathematical arguments during discussions. In order to incorporate both arguing and connecting as being at the heart of mathematical discussions and to highlight extending student thinking as an overarching umbrella, I propose that the Connecting practice is elaborated into the *Extending by arguing and connecting* practice.

The teacher’s role in whole-class discussions is to build upon students’ reasoning about their ideas and to help them advance key mathematical ideas and connections in order to create opportunities for them to extend their thinking. In this, the teacher needs to promote further reflection and arguments from the students (Ruthven et al., 2011). To be able to recognize moments in whole-class discussions that create possibilities for extending student thinking by arguing and connecting, teachers need to be well-prepared. With the powerful help of working with the preceding practices of Stein et al.’s model the teacher can prepare for the arguing aspect to a certain extent in advance, as is also the case for connecting. Thus, the practice of *Extending by arguing and connecting* is in line with the strong emphasis on planning in Stein et al.’s model.

Discussion

From my collaboration projects with in-service teachers, I have found three dimensions along which to elaborate on Stein et al.’s model: breadth, depth and length. The suggestion to broaden the last practice to also include arguing falls into the first dimension. A suggestion that falls into the second dimension is to deepen the last practice to distinguish between different kinds of connections. Connections can for example be made between representations (Cengiz et al., 2011), especially between different forms of representations, between solution strategies (Stein et al., 2008) and between lessons or units (Cengiz et al., 2011; Lampert, 2001). Ma (1999) distinguishes between connections to basic ideas (concepts and principles) and connections between multiple approaches of an idea. The suggestion to deepen the connecting practice is based on my observations during several intervention projects that teachers who are new to the approach of teaching mathematics through problem-solving may make limited connections (Larsson & Ryve, 2011; 2012). Finally, a suggestion that falls into the third dimension, made by teachers in my two-year intervention project, is to lengthen the model with a launching practice in which the teacher leads a whole-class discussion to introduce the problem in order to address the issue of equity properly (Jackson & Cobb, 2010).

I conclude with discussing the practical implications for in-service and pre-service teacher education. Working with a tool such as Stein et al.'s model has the potential of helping teachers over time to conduct mathematical discussions that focus on important relationships between mathematical ideas, which is a key ingredient in high-quality teaching that aims at developing students' mathematical competencies. I have used Stein et al.'s model extensively in both in-service and pre-service education and many teachers express that their whole-class discussions are raised to a new level with the help of the model. My suggestions to elaborate on Stein et al.'s model to also incorporate the practice of arguing and to refine the sequencing practice can make the model even more useful to teachers. My work will continue with elaborating on Stein et al.'s model and also to further explore the moment-to-moment decisions during the classroom interaction that a mathematics teacher need to take in order to promote students' further reflection and arguments.

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Inconsistency, Regression or Development? The Professional Identity of a Novice Primary School Mathematics Teacher

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There is an increasing awareness of the social dimensions in the professional identity development of mathematics teachers. This paper reports on similarities and differences in how a novice teacher talks about good mathematics teaching and high-performing mathematics students at the time of her graduation and then one year later. By analysing the social dimensions of the novice teachers' professional identity development these changes, often referred to as inconsistency and/or regression, can be understood as development in her memberships in different kind of communities of practice.

Introduction

The teaching profession, with or without focus on mathematics teaching, is often described in terms of a changed profession without much continuity between teacher education and schools (Cooney, 2001; Sowder, 2007). Several studies report that what novice teachers of mathematics have learned in teacher education tends to regress when they start work as teachers (Bjerneby Häll, 2006; Cooney, 2001; Sowder, 2007). In contrast to teacher education, novice teachers' own schooling is often attributed an important value in relation to how student teachers and novice teachers think about teaching and how they teach (Gellert, 2000; Lortie, 1975; Persson, 2009; Wang, Odell & Schwille, 2008).

Many previous studies regarding becoming a mathematics teacher have focused on student teachers' and/or novice teachers' beliefs. In several of these studies teachers' appear to be inconsistent towards their beliefs (Phillip, 2007). This is explained in different ways, for example that beliefs are situated, that different beliefs are dominant in different situations, that the individuals has unconscious beliefs or that the researcher and the teachers have different interpretations of concepts (Goldin, 2002; Phillip, 2007; Speer, 2005; Wilson & Cooney, 2002). However, Phillip (2007), Speer (2005) and Wilson and Cooney (2002) all stress it as problematic when researchers claim teachers to be inconsistent and according to Phillip (2007) inconsistency stop existing when researchers better understand the teachers in relation to their social environment.

In recent years research on teachers' professional identity formation has expanded (Beijaard, Meijer & Verloop, 2004; Ponte & Chapman, 2008). Graduating from teacher education and starting to work as a teacher can be seen as a transfer or shift in professional identity where the interplay between the individual and their social environment is highlighted as a central part about which to develop understanding (McNally, Blake, Corbin & Gray, 2008). Studies of professional identity consider not only what teachers know and/or believe but also who they are, how they view themselves as teachers, how they relate to students, how they deal with problems, how they reflect on issues, and how they identify themselves with the profession. Important, too, are their relations with parents and colleagues, their participations in professional groups and the kind of teacher they want to be (Ponte & Chapman, 2008)

The empirical material presented in this paper derives from a study of novice primary school mathematics teachers' professional identity development (Palmér, 2013). In this paper the focus will be on how one novice primary school mathematics teacher, Nina, talks about good mathematics teaching and high-performing [1] mathematics students at the time of her graduation and then one year after. In the case of Nina there are both similarities and differences at the two times focused on. The question to be investigated in the paper is if the differences are to be understood as inconsistency, regression or development.

Professional identity development

Peressini, Borko, Romagnano, Knuth and Willis (2004) argue for using a situated perspective in studies of mathematics teachers' teaching. The term situated refers to a set of theoretical perspectives which conceptualise learning as changes in participation in socially organised activities and individuals' use of knowledge as an aspect of their participation in social practices.

In this paper a situated perspective, communities of practice (Wenger 1998), is used aiming to capture both the individual and the social dimensions of professional identity development. A community of practice is defined through the three dimensions of mutual engagement, joint enterprise and shared repertoire. Mutual engagement is the relationships between the members, about them doing things together as well as negotiating the meaning within the community of practice. Joint enterprise regards the mutual accountability the members feel in relation to the community of practice and it is built by the mutual engagement. The shared repertoire in a community of practice regards its collective stories, artefacts, notions and actions as reifications of the mutual engagement.

According to Wenger (1998), identity formation is a complementary dual process in which one half is the identification in communities of practice and the other half the negotiation of the meaning (regarding the mutual engagement, joint enterprise and shared repertoire) in communities of practice. An individual can identify and negotiate in communities of practice through engagement, imagination and/or alignment (modes of belonging). Engagement implies active involvement and requires the possibility to physical participation in activities. Imagination implies going beyond time and space in physical sense and create images of the world and makes it possible to feel connected even to people we have never met but that in some way match our own patterns of actions. Participation through alignment implies that the individual change, align, in relation to the community of practice the individual wants to, or is forced to, be a member of. These three ways of identifying and negotiating involve different approaches and different conditions and do not require or exclude each other. Since imagination and alignment expand participation in communities of practice beyond time and space in physical sense individuals can be members of and sense belonging to communities of practice without visible shared practice.

The study

Nina is 24 years old when she is about to graduate from teacher education. She is specialised in science, technology and mathematics for primary school. Within her teacher education she has taken 37,5 credits of courses in mathematics education.

The empirical material in this paper is from the first year after Nina's graduation from teacher education. An ethnographic approach has been used to make visible the process of professional identity development in communities of practice. Ethnography is not a collection of methods but a special way to look at, listen to and think about social phenomena where the main interest is to understand the meaning activities have for individuals and how individuals understand themselves and others (Arvatson & Ehn 2009; Aspers 2007; Hammersley & Atkinsson 2007). According to Aspers (2007), gaining such an understanding requires interaction which implies that the researcher participates with, observes and interviews respondents in the field of study.

The empirical material in the case of Nina is from self-recordings made by her, observations and interviews. All of these have been made in a selective intermittent way (Jeffrey & Troman 2004) which means that the time from the start to the end of the fieldwork has been long but with a flexible frequency of field visits. To accomplish a balance between an inside and outside perspective in line with the ethnographic approach (Aspers, 2007); the observations have been both participating and non-participating. For the same purpose the interviews have been both spontaneous conversations during observations and formal interviews (individual and in groups) based on thematic interview guides. The self-recordings were recorded by Nina herself on an mp3-player. She was told to record whatever and whenever she wanted and that it was up to her to decide what was important for the researcher to know about starting to work as a primary school teacher of mathematics.

These varying empirical materials (observations, interviews, self-recordings) have different characteristics but are in the analysis treated as complete-empiricism (Aspers, 2007). In this paper only how Nina's talks, not how she acts, is focused on. However, the analysis of her talk is based on the complete empiricism implying all the empirical material constituting wholeness. Based on this complete-empiricism interpretations are made regarding her engagement, imagination and/or alignment in different communities of practice she seems to negotiate and/or identify with and how these memberships influence her talk about about good mathematics teaching and high-performing mathematics students at the time of her graduation and then one year after.

The case of Nina

In this section the case of Nina will be presented in three sub-sections. In the first sub-section the time of her graduation will be focused on. In the second sub-section the time one year after her graduation will be focused on. The joint theme in these two sections is how Nina talks about good mathematics teaching and high-performing mathematics students. In the third sub-section similarities and differences in her talk at the two times are focused on.

Nina at the time of graduation

The first interview with Nina is conducted three weeks before her graduation from teacher education. In the interview Nina says that she has experienced a "new approach" to mathematics teaching during her teacher education and she expresses a very clear opinion regarding how she wants to "reform mathematics teaching". When being asked to give examples of good mathematics lessons she tells about lessons "outside the frames" of the text book, for example:

> We worked with the number eight. And then we played bowling with the children. And it really is an example, a concrete example, they didn't think much of it as mathematics but they counted the whole time, how many fell and were left standing. And the whole time they saw the connection to eight. (interview)

The examples Nina gives of good mathematics lessons can be summarised as varied, laboratory-based, concrete, reality-related and problem-orientated. As good, she also emphasises mathematics teaching where the students do not realise that they are being taught mathematics. Such mathematics teaching is, according to Nina, student-centred and captures the students' interest. Nina distinguishes between this approach to mathematics teaching and her own experiences as a student in school and the teaching she has met during pre-service teaching.

> [I have] been at two different schools quite a long time and it feels like many teachers are very controlled by the text book and that is what counts (interview).

The good examples Nina gives are from teacher education and her own teaching during practice periods. When talking about these mathematics lessons she refers to “we” as in herself and fellow students from the teacher education.

Further Nina talks about “stimulating” all students in a mathematics class, not only “the norm in the class” but also the “weak and the strong” students. She says that her examples of good mathematics teaching “refers to all children”, both the “weak and the strong”. She specially emphasises the importance of paying attention to and challenge the “high-performing” mathematics students.

Less good mathematics teaching is, according to Nina, “old-fashioned”, “traditional”, following a “patterned scheme” within the “frames” of the text book where the students do not cooperate and solve tasks in only one way. She says that a strictly use of a text books can result in an incorrect interpretation of the fast students as being the “high-performing” ones, while the ones that really are the “high-performing” do not get any input except “sit like that and work in their text book”.

Nina one year after graduation

After graduation Nina moves back to her hometown and during the following year she seldom has contact with her fellow students. At this time it is difficult to get an employment as a primary school teacher in Sweden since there are more educated teachers than teacher jobs. In the absence of teacher jobs Nina starts to work as a teacher assistant at Aston School for John, a boy in grade one, who has attention deficit hyperactivity disorder. Aston School has three classes in every grade from preschool class up to grade six. Nina likes Aston School but her work as a teacher assistant (spending all day with John) prevents her from joining the fellowship with the other teachers except John’s class teacher Diana. Nina say’s that Diana is as a “tutor” for her and that they are “very close”. Except Diana Nina does not cooperate with any of the other teachers at Aston School and she describes herself as the “lonely one”.

> Diana and I use each other to get things done. And all the time we are two resources which the other teachers are not. […] The only thing is that I don’t have time for planning and therefore I never attend any meetings with the other teachers, conferences about students or anything. Because of that, I don’t really belong to any staff group. (self-recording)

At Aston school they work with ability groups when teaching Swedish and mathematics. Nina says that this organisation works out fine and that the groups focus on totally different things.

> The group containing the slightly weaker students' moves along very slowly, they do very simply tasks [...] Then the groups with students who are good and interested in mathematics, if you can say it like that, they work faster, moving forward. They don't have to keep the group together; everyone works in their own direction. Everyone does different tasks [in the text book]. You are simply left to work at your own pace and to become good at what you want. (self-recording)

During the mathematics lessons John is in the ability group with "good and interested" students, which is taught by Diana. When talking about the mathematics teaching in this group, Nina says "our mathematics teaching" and "our class". Since Nina has no time for planning it is the Diana who plans the mathematics lessons. The lessons are based on a text book that Nina says that she "actually" likes. She says that the text book is different from "the ordinary ones she counted in when she was little". As good with the text book, she stresses that every chapter starts with the goals for that chapter followed by a "math lab" where the students work with "practical material" in pairs "showing what they have done and limn each other's solutions". According to Nina, this is good since the students "are to see how differently they think and that it can be right irrespectively of how they have done it". However, as the group of "good and interested" students work in their own pace it is not always possible for them to work together.

Similarities and differences at the time of graduation and one year later

When comparing how Nina talks about good mathematics teaching and high-performing mathematics students at the time of her graduation and then one year later there are both similarities and differences. One example can be seen in how she talks about text books. Just before graduation she expressed mathematics teaching based on the text book as "old-fashioned" and negative. One year later she "actually" likes the text book. However, the words used to describe why she likes the text book are similar to the words she used to describe good mathematics teaching just before graduation. Just before graduation she expressed good mathematics teaching as varied, laboratory-based, concrete, reality-related and problem-orientated. One year later she expresses the text book as good because it includes the use of practical material, math lab, and work in pairs where the students are to show their different solution.

Another example containing both similarities and differences is how Nina talks about student's different levels in mathematics. Just before graduation, she talked about the importance of teaching every student on their level which is in line with the ability groups used at the Aston school. However, before graduating, she stressed that the fast students are not necessarily the ones who are high-performing and, that the high-performing students need challenges other than working in the text book. After one year, when she talks positively about the ability groups used at Aston school, the pace of working in the text book is central where the “slightly weaker students” work slowly and the “good and interested” students are to work individually, in their own pace, in their text books.

Analysis and Discussion

How are these similarities and differences to be understood? As shown in the introduction own schooling is often attributed an important value in relation to how teachers think about teaching and how they teach. Further studies have shown that what novice teachers have learned in teacher education tends to regress when they start to work as teachers. Based on such studies one explanation could be that Nina has regressed and now emphasise the “traditional” individual text book centered mathematics teaching she herself experienced as a student as good. Based on the empirical material in the case of Nina (which does not include Nina’s time as a student in primary school) no interpretations can be made regarding how her talk one year after graduation equals the mathematics teaching she herself has experienced as a student. However, when she says that the text book used at Aston school is good she says that it differs from “the ordinary ones she counted in when she was little”.

As also shown in the introduction an explanation based on beliefs research could be that Nina is inconsistent in her talk at the two times. However, Phillip (2007), Wilson and Cooney (2002) and Speer (2005) all stress it as problematic when researchers claim teachers to be inconsistent and according to Phillip (2007) inconsistency stop existing when researchers better understand the teachers in relation to their social environment.

Instead, in this paper, Nina’s talk about about good mathematics teaching and high-performing mathematics students at the time of her graduation and then one year after will be analysed in relation to her memberships in forms of engagement, imagination and/or alignment in different communities of practice she seems to negotiate and/or identify with. Maybe she, based on her experiences the year after graduation, has developed a new view regarding text books and high-performing mathematics students. As mentioned, this analysis is based on the complete empiricism implying that the analysis of her talk one year after graduation is based on all the empirical material in her case (interviews, self-recordings and observations).

Nina’s descriptions of good, and less good, mathematics teaching at the time of her graduation can be understood as her having a membership in a community of reform [2] mathematics teaching. In this community of practice, there is a joint enterprise and a shared repertoire regarding good and less good mathematics teaching. At the time for graduation Nina participates in this community of practice by engagement and imagination as imagining her future teaching. As for engagement, Nina does not express being a part of the negotiation of the shared repertoire, but she has been engaged in its teaching during her teacher education.

One year later two communities of practice are visible in Nina's talk about mathematics teaching. One is the above described community of reform mathematics teaching. The possibilities for Nina to participate by engagement in this community disappeared when she graduated from teacher education and moved away from her fellow students. One year after graduation she participates mainly by imagination and she does not carry out any mathematics teaching in line with its shared repertoire. The new community of practice is a community of teachers working in John's class, that is Nina and the class teacher Diana [3]. Based on her work as teacher assistant this is the only community of teachers that Nina can participate in at Aston School but she does not express any kind of alignment. Together Diana and Nina work with the high-performing mathematics students at Aston School. Even if Nina is not involved in the planning of the mathematics lessons she talks about "our mathematics teaching" and "our class". Diana is the core member in this community through planning and shaping its shared repertoire and Nina participates by engagement.

Nina's talk about good mathematics teaching and high-performing students one year after graduation seems to be a merger of the shared repertoires in the community of reform mathematics teaching and the community of teachers working in John's class. In "our mathematics teaching" the text book is the core role and Nina says that she "actually" likes it. This "actually" can be related to the negative role of the text book in the shared repertoire in the community of reform mathematics teaching. Further Nina evaluates the text book centred teaching in relation to the shared repertoire in the community of reform mathematics teaching (practical material, math lab, work in pairs where, different solution). In the community of reform mathematics teaching it is important to take both the "weak and the strong" students into consideration in the mathematics teaching. One year after graduation Nina still emphasise this but who the high-performing students are and the strategy for considering their needs has changed. In the community of reform mathematics teaching the high-performing students are not necessarily the fast students while in the community of teachers working in John's class the pace of working in the text book is central. Further, in the community of reform mathematics teaching the high-performing students need other challenges than working in their text books while in the community of teachers working in John's class the high-performing students work individually in their text books.

Conclusion and Implications

Before graduation Nina expressed a clear opinion regarding good mathematics teaching and high-performing mathematics students. One year later there are both similarities and differences in how she talks about the same issues. The differences in her talk that may look as inconsistence or regression in the eyes of an observer becomes consistent when analysing the social dimensions of her professional identity development. By analysing the social dimensions of her professional identity development her talk one year after graduation can be described as her merged participation in two different communities of practice. Some might argue that it was previous known that school culture and colleagues impact novice teachers. However, the results presented in this paper enable an understanding of how such impact evolves. Furthermore, this understanding makes it possible to reinterpret earlier studies presenting novice teachers changes as inconsistence or regression. Maybe, it is not inconsistence or regression, but professional identity development as new or increased memberships in communities of practice regarding mathematics teaching.

Notes

1. Nina alternates between the terms ”high-performing”, ”gifted” and ”good and interested”. When she is not quoted the term ”high-performing” will be used for consistency.
2. The term “reform” used for this community of practice is based on Nina’s use of the word in relation to her description of good mathematics teaching.
3. Diana may be part of a larger community of teachers at Aston school but from Nina’s perspective it is only Diana who she is involved working with.

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When the Mathematics Gets Lost in the Teaching

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This study shows how six elementary teachers, construed locally as effective, interpreted and were observed to enact the same curricular and didactical language very differently. One group of three provided high-level cognitively challenging tasks to engage children in mathematics. A second group of three, ensuring their children enjoy mathematics, subordinated mathematical learning to an emphasis on the teaching activity. The actions of this second group made mathematics invisible.

Introduction

The study looked in-depth not only at how six teachers taught mathematics but, more importantly, justified the ways in which they presented the subject. The results were both surprising and cautionary. This paper reports on how the mathematical learning intentions of a teacher can get lost in the attention paid to the teaching activities employed. All teachers espoused a rich problem-solving environment, but in reality, the manifestations of their beliefs varied greatly. Why?

Drawing on the traditions of grounded theory from which the constant comparison process derives (Glaser and Strauss, 1967), I will now present the methodology and methods in order to clarify the approach before introducing the literature pertaining to the categories identified in the data.

Methodology & Methods

Case study allows us to explore in-depth how and why teachers teach in the ways they do (Silver, 2013). To this end, a multiple exploratory case study (Stake, 2002) was undertaken to examine six elementary teachers' perspectives on, and justifications for, the mathematics they expect their children to learn. Each teacher, who had specialised in mathematics during training, was well-qualified, considered locally to be effective and, importantly, an ambassador for the subject. This purpose sampling (Denzin & Lincoln, 2011) was intended to avoid the dichotomisations typically found when generalists are compared with specialists, particularly from the perspective of confidence (Goulding, Rowland, & Barber, 2002; Peker & Erekin, 2011).

Three approaches to data collection were employed to optimise the likelihood of unravelling the relationship between espoused belief and enacted practice. Initial interviews explored teachers' perspectives on the nature of mathematics and its teaching; video-recordings of random lessons, typically four per teacher, yielded evidence of patterns of practice and highlighted teachers' mathematical emphases; stimulated recall interviews (SRI) conducted shortly after each lesson elicited teacher's espoused intentions and justifications for their actions.

As with most case study investigations, much data was collected and, as is explained below, existing theoretical and analytical frameworks proved inadequate for meaningful interpretation. For example, a comprehensive framework for analysing teachers' teaching activities and inferable learning outcomes, used in an earlier comparative video study (Andrews, 2007), was unable to capture the complexity of the belief-practice relationship. Other frameworks, for example, Askew, Brown, Rhodes, Johnson, & William's (1997) categorisation of teacher types or Kilpatrick, Swafford, & Findell's (2002) strands of mathematical proficiency, while able to support elements of the analysis, proved too lacking in specificity to be useful, even when employed in combination, highlighting the elusive and unpredictable nature of the belief-practice relationship (Skott, 2004).

These difficulties led me to adopt the constant comparison analytical approach of grounded theory (Strauss & Corbin, 1998), which is commonly used in case study (Yin, 2009), as it facilitates the thick description expected of case study analyses of complex educational settings (Merriam, 1998). In brief, constant comparison in this context entailed a repeated reading of the data from the first case to identify categories of belief and practice. As each was identified, the case data were reread to see if had been missed earlier. On completion of this first pass a second case was read for evidence of both the earlier categories and new ones. As each new category was identified, all previous case material was scrutinised again. Categorical definitions constantly refined as incidents were compared and contrasted (Denzin & Lincoln, 2011). This process of continual comparison and refinement, which facilitates the integration of categories into a coherent explanatory model (Taylor & Bogdan, 1998), led to the identification of five categories of teaching activity common to all five teachers: *exploiting prior knowledge; creating mathematical connections; using mathematical vocabulary; encouraging mathematical reasoning and exploiting rich mathematical tasks.*

Results & Analysis

In the following I examine briefly the literature related to each category of teaching activity before presenting and discussing how they played out in the beliefs and practices of the six case study teachers, here given the pseudonyms Caz, Ellie, Fiona, Gary Louise and Sarah. In their interviews, all six made strong reference to each of the five categories although the manifestation of those beliefs varied considerably. The teacher utterances are presented in italics below together with thick descriptions to illustrate the consistent emphasis made by the teachers in these five categories.

Exploiting prior knowledge

The activation of students' prior knowledge has long been associated with constructivist perspectives on learning whereby "information is retained and understood through elaboration and construction of connections between prior knowledge and new knowledge" (Kramarski, Mevarech, & Lieberman, 2001: 298). Indeed, there is evidence that the more effective teachers are in their activation of students' prior knowledge, the more profound the student learning (Kramarski et al 2001). Research has also shown that while a student's prior knowledge is a strong precursor of new learning, when combined with student interest, the effect was greater (Tobias, 1994). The ability of teachers to activate students' prior knowledge is a strong indicator of the quality of a teacher pedagogical content knowledge (Baumert, Kunter, Blum, Brunner, Voss, & Jordan, 2010).

Data analysis

Although all teachers were seen to emphasise *prior mathematical knowledge* at the beginning of each lesson, there were differences in their justifications for so doing. For Sarah, Fiona and Gary, the first step of every lesson was to bring to mind what the children had been learning previously. Where children failed to respond to direct questions they reminded them about activities they had undertaken together, e.g. Gary said '*remember when we had that polling booth in the classroom for the American elections*?' All three asked closed and tightly focused questions expecting a single correct answer. Interviews revealed that all three saw this as a linear 'stepped-process' within their lesson structure.

In contrast Caz, Louise and Ellie gave their children time to think and talk to partners about what they remembered or what they knew about the question asked. Moreover, this giving of time occurred, whenever an issue or idea appropriate for discussion arose. All three offered precise reasons related to giving children time to think mathematically. Caz based her understanding of child psychology training in how both children's understanding and mathematical concepts are built upon previous material.

Creating mathematical connections

There is increasing evidence that where teachers make an appropriate and explicit *connection* between the mathematical concepts and procedures they teach, students acquire a more profound understanding of the subject and are able to solve more complex problems (Askew et al.,1997; Schneider & Stern, 2010). That is, where teachers encourage a relational view of mathematics - an understanding of structural relationships within and between concepts - rather than an instrumental view - rules characterised by mechanical steps - learning is deeper and made applicable (Skemp, 1987). However, if connections are encouraged inappropriately then the intended mathematics may not emerge, as Van Zoest & Bohl (2005). They found that some teachers create connections between and within their teaching activities rather than between and within mathematical entities.

Data analysis

Caz and Louise seemed to fit the description of a connectionist teacher (Askew et al.1997) well through illustrations or modelling explanations. They made explicit connections between different elements or concepts of mathematics. E.g. Caz was observed to hold a marked (counting) stick horizontally to model a number line before turning it through 90° and describing it as a scale. She believed that such representations *help children read scales...like... on a thermometer... particularly when the scale on the 'Y' axis does not represent one* unit. She was aware too, of avoiding colluding in the construction of children's misconceptions. She commented that a *common mistake children make is assuming each line up the y-axis is one, so I do not always count in ones on the counting stick.* In such actions Caz responded to both her perception of the children's needs and her ambition to *take them a little further on and make that connection*.

Ellie rarely made explicit connections, although she provided opportunities for children to make them for themselves. On one occasion a boy described a quadrilateral as a '*truncated triangle'*. Ellie later explained that this particular boy had been exploring solids the week before and, having spotted a truncated cone, wanted to know what it was called. Ellie encouraged children to develop both enthusiasm and a sense of enquiry. Interestingly, Ellie offers an alternative view on Askew et al.'s (1997) connectionist teachers, as she did not prompt explicit connections but did so implicitly.

Fiona made no explicit connections between areas of mathematics, but exploited concrete materials to illustrate concepts; for example plastic linking cubes were used to illustrate the partitioning of two digit numbers. However, observations highlighted some confused children as her vocabulary of *big ones* (tens) *and little ones* conflicted with the place value cards (20 and 5 for 25), she had used earlier. During interview she stated that, for her, it was not an issue, having 'told' her children how the concrete materials were connected to the concept of place value, so she would just repeat this learning again in the term. This particular event seemed indicative of a lack of awareness of the impact of her actions on her children's understanding of place value. Interestingly, Fiona consistently emphasised her role of *telling* of concepts to children.

Throughout their lessons, both Sarah and Gary made explicit connections between activities, rather than the mathematical concepts embedded in them. For example, Gary spoke about the ways in which his class collected data during a mock poll related to the US Presidential vote but not about the data themselves. Sarah used many manipulatives e.g. use of coloured cards, making explicit connections between the use of the cards rather than the concept being taught. That is, in the mind of these teachers the connection was made to the mathematics, but in reality the connection was made to the activity or context and not the mathematics, just as Van Zoest & Bohl, (2005) had found.

Using mathematical vocabulary

Being mathematically proficient means that one must acquire, understand and use effectively an appropriate vocabulary (Barwell, 2005). However, the acquisition of such a vocabulary is complex. As Steele (1999) notes,

> "Children develop language through their experiences. They develop, clarify, and generalize meanings of words by learning the words as symbols of experienced concepts, using the words, and having the people around them react to their word use. (Steele, 1999: 39)

This need to react to students' word use creates problems for teachers (Watson & Mason, 2007) not least because inducting students into an appropriately understood and operational mathematical vocabulary is typically a consequence of a guided interplay between formal and informal language, Leung (2005).

Data analysis

Louise, Ellie and Caz frequently used games to encourage children's use of new and unfamiliar mathematical vocabulary. Sarah, Gary and Fiona provided lists of words, expecting children to use them in response to closed questions and would frequently read out these words during their lessons. Such practices, it seems, highlight von Glaserfeld's (1991) distinction between *teaching* children and *training* children. He adds that teachers have a better chance to modify children's' conceptual structures if a model informs interventions, such as the opportunity to use new vocabulary naturally, such as in a game.

Encouraging mathematical reasoning

The development of students' mathematical reasoning is key objective of mathematics education (Hill, Ball, & Schilling, 2008). However, traditional teaching typically fails to encourage long term gains due to emphases on superficial memorisation strategies rather than the mathematical properties under scrutiny (Lithner, 2000). Indeed, a teaching emphasis on worked examples is inferior to the encouragement of metacognitive training in facilitating students' mathematical reasoning (Mevarech & Kramarski, 2003). Such matters are strongly linked to notions of teachers' mathematical knowledge for teaching, not least because mathematical reasoning is a much higher order activity than the conceptual and procedural knowledge dominant in most classrooms (Rowland & Ruthven, 2011).

Data analysis

Expectations that children would think mathematically and engage in *reasoning* were consistently observed throughout Louise, Ellie and Caz's lessons. Caz encouraged children to 'argue' with her if they were confused or disagreed with anything she said. Often evoking such argumentation purposefully. During one fractions-related episode she had failed to notice an ambiguity in her presentation of a problem. It went, *if there were two cakes and six people*, *how many pieces would each person have?* One child, William, said that they will have one sixth of one bar and one sixth of the other before concluding, that each person would have two sixths altogether. Another child, Holly, pointed out that it should be two-twelfths not two sixths. This created a lengthy discussion amongst the class and although some children had accepted Holly's explanation, Caz explained later that she was eager to discuss the cognitive conflict to demonstrate how fractions can be confusing. Unpacking the problem as it arose (critical incident) and working with the children in reconciling the two perspectives was very much Caz's reflection of the incident.

Fiona did not emphasise reasoning or thinking in any discussion we had. Her focus was on her teaching activities rather than mathematical learning. This was an interesting observation as her explanations, like Sarah's, were to focus on the '*how to do...*' something rather than what it is connected to, or *why* they were learning this element of mathematics, other than it was an *assessment target*. In similar vein, Gary's focus was the acquisition of knowledge necessary for passing statutory tests the following year, which, were manifested in his frequent use of mathematical memorising exercises of facts. Indeed, Gary was adamant with respect to the importance of such practices in mathematical learning, often emphasising the role of tricks, practising and the memorising of facts, just as when he was a child at school.

Rich mathematical tasks

Mathematical tasks play a key role in facilitating understanding and discussion (Stein et al, 2014). The more 'complex', 'worthwhile' and 'intellectually-challenging the task, the more likely students are to acquire not only higher order knowledge and skills but also positive dispositions towards the subject (Silver et al., 2013). Teachers' use of rich tasks is typically construed as reflecting high expectations for student learning (Kazzemi & Franke, 2004).

Data analysis

Louise, Caz and Ellie demonstrated an understanding of where the concepts they were teaching would lead and chose specific tasks as a consequence. These were not always planned for, and often a consequence of critical incidents (Cooney, 1987). For the remaining three teachers, activities were drawn from a series of photocopiable teacher resource books, or a snapshot of different concepts jumbled into one lesson, with little emphasis on related learning, concepts or mathematical intent. For example, although both Caz and Gary discussed real-life tasks during interview, the ways in which these were presented differed starkly. Caz tended to draw on her *children's* real life experiences to illustrate or reinforce a concept. E.g. she emphasised the irregularities in people's abilities to reference the passing of time by asking children to identify aspects of their lives related to the notion of five minutes. This led to her commenting, in interview, that *for Latia it was about mathematics in dancing, for Josh it was about swimming the length of a pool and for Tom it was about scoring a goal*. Caz used such serendipitous moments and real-life experiences to encourage children to think mathematically. In contrast, Gary also referred to real-life situations but *directed* his children to specific events like the American presidential elections they had previously modelled in class. His justifications were similar to those of Caz, drawing on the importance of real-life situations, but the difference was that Gary provided both content and context. Thus, he made all the thinking, and connections.

Discussion

All six teachers were aware of the relevance of the five components to mathematical learning. However, the classroom manifestations of similarly espoused beliefs tended to dichotomise. On the one hand was a group, Caz, Ellie and Louise, whose beliefs and practices were commensurate in their explicit focus on children's learning of mathematics. On the other hand was a group, Sarah, Fiona and Gary, whose beliefs, while clearly located in the same vocabulary as the first group, focused on issues independently of the mathematics they may or may not have taught; a group whose practices subordinated mathematics to teaching activities. For example, while all focused on the activation of prior knowledge at the start of their lessons, these three believed and behaved as though it were a ritual element of all lessons - *talk about what we did last time and then move on*.

One group talked about the activities they employed, independently of the mathematics they taught, while the latter focused explicitly on mathematical ideas. That is, the one group referred to the enjoyment of learning, irrespective of mathematics, while the other referred to the challenge that is mathematics (Moyer, 2001). In other words, one group seemed focused on *training* children, while the other on *teaching* children (Von Glasersfeld, 1991). This notion of training was clearly reflected in Gary's encouraging his children to use a vocabulary list to answer his questions. The mathematics also appeared to get lost in Sarah and Fiona's class, as they both focussed on the '*how to do...*' something as the means of addressing their next *assessment target*. In sum, Gary, Sarah and Fiona's practice presented very few opportunities for children to engage meaningfully in mathematical reasoning. They believed they did, but observations indicated that this was subordinated to enjoyment. For them, mathematics was about how they taught; it was not about the cognitive engagement of children in mathematics.

Conclusions

When I started this study, such differences in experienced specialist teachers' mathematical objectives were unexpected, as all were well qualified, and acknowledged locally as effective. Yet, only three of the six teachers provided consistent opportunities for children to think and explore collectively while making connections with and for each other individually. Explicit collective construction of new mathematical knowledge was privileged, by means of rich tasks, individual enquiry, argumentation and justification supported by an expectation of appropriate mathematical language. Their three colleagues consistently attended to how rather than what they taught - their attention was on activities, manipulatives, incremental steps and amount of mathematics covered. For one group the mathematics was transparent and for the other it was opaque, warranting the question, *Where is the mathematics?*

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Designing tasks and finding strategies for promoting student-to-student interaction

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To reach the goals of communication and reasoning in mathematics in upper secondary school, students need to talk about mathematics but sometimes this is not as easy to achieve as it first seems. In this paper, an initial analysis is provided of tasks and strategies from an educational design research project promoting student-to-student interaction. The data include students' interactions and perceptions on working with mathematics in groups from the first of three cycles. They are analysed and discussed in relationship to the choice of analytical tools, means of support and tasks for the remaining cycles.

Introduction

Skolinspektionen, the Swedish School Inspection Department (2010), criticized the fact that in many upper secondary mathematics classrooms, students do too much individual work in textbooks. The introduction of a new syllabus in mathematics in 2011 (Skolverket (the Swedish National Agency for Education), 2012) increased the focus on communication and reasoning abilities. Consequently, there is a need to investigate how to achieve this focus. In my larger research project, tasks are introduced which were designed to improve students' mathematical communication abilities. The research questions are: How do interactions and perceptions change over time when different tasks are provided to increase student-to-student interaction? What kind of strategies and tasks promote student-to-student interaction? Here the first question focuses on students' perspectives, while the second is on the pedagogical choices made in the project. In this paper, an analysis of the implementation of the first set of tasks is described in regard to the implications for further tasks.

The study is conducted in a first year, upper secondary classroom in a city in Sweden. The teacher was interested in trying new strategies concerning student interaction. Almost all students in the class have foreign backgrounds, which according to Skolverket (2013), means that they were born abroad or born in Sweden with both parents born abroad. Since almost one quarter of all students in Swedish upper secondary schools have foreign backgrounds (Skolverket, 2013), it is common that at least some students in a classroom do not have Swedish as their first language. Van Eerde, Hajer and Prenger (2008) claimed that second language learners "need to actively use and produce new linguistic elements" (p. 34). However, this may also be the case for first language speakers, since it is crucial for all mathematics students to explain, reason and justify (Brandt & Schütte, 2010).

Background to the study

In order to study increased student-to-student interaction, this project uses educational design research (EDR) (McKenney & Reeves, 2012). EDR allows for tasks to be designed flexibly and supports ongoing changes in teaching practices. EDR is a cyclic process in which each cycle contains three phases: analysis/exploration, design and evaluation (McKenney & Reeves, 2012). Working through the phases provides opportunities for improving the tasks but also for producing theoretical understandings (McKenney & Reeves, 2012; Van den Akker, Gravemeijer, McKenney & Nieveen, 2006). In this project, the focus is on developing theory on student-to-student interaction, while developing a practical intervention.

The project consists of three design cycles that have mathematical as well as student interaction goals. In EDR, the choices made in each cycle need to be theoretically justified (McKenney & Reeves, 2012). Thus in this case, the designs are developed from theories on interaction and mathematical communication. Therefore, design means not only the tasks given to the students, but also the means of support for student-to-student interaction and the organisation for lessons in which the tasks are implemented.

Since research concerning student-to-student interaction in multilingual upper secondary mathematics classrooms appears to be limited (see Goos, Galbraith & Renshaw, 2002; Forster & Taylor, 2003), theories are drawn from research with younger students or in monolingual settings. A starting point has been theories on cooperative learning, which is a family of methods in which students learn in small groups and take responsibility for each other's learning (Brandell & Backlund, 2011). In cooperative learning, it is important that there is a positive interdependence between the students, so that the students have the common goal of solving tasks together. To be successful with the tasks, all students need to succeed. Walshaw and Anthony (2008) claimed that group work gives students opportunities to express their thinking and that "small group work can provide the context for social and cognitive engagement" (p. 142).

However, not all group work is effective. Sfard and Kieran (2001) provided an example of an unsuccessful collaboration and concluded that just because students talk, it does not mean that they learn. Another example is Fuentes's (2013) action research project that identified issues preventing effective communication, such as the promotion of communication, the quality of the communication and socio-cultural norms (Fuentes, 2013).

In order to overcome some of the difficulties identified with group work, Alrø and Skovsmose's (2004) inquiry cooperation model (IC-model) was used as a theoretical base for the first cycle. It describes how to create opportunities for rich conversations about mathematics. This model usually concerns teacher-student mathematical communication, but in this study was applied to student-to-student communication. It allowed student interactions to be analysed by the type of communication acts about the mathematical tasks. The communicative acts were: *getting in contact*, *locating*, *identifying*, *advocating*, *thinking aloud*, *reformulating*, *challenging* and *evaluating*. Although, Alrø and Skovsmose (2004) claimed that it is not common to find fully developed IC-models in classrooms, it seemed a valuable way of understanding how the students interacted together to solve mathematical tasks.

In this project, three cycles are conducted during one semester. Students are audio-recorded while working with the tasks. They also complete a questionnaire and are interviewed in groups of two to four students after each cycle.

The first design cycle

Goals

In the first design cycle, the mathematical goal was to develop students' mathematical problem-solving strategies. Problem solving is a part of mathematics in which the answer and/or the solution methods are not directly apparent to the students (Schoenfeld, 1983). Initially this had seemed an appropriate context for encouraging the students to discuss mathematics with each other. The goal concerning group work and communication was that all students would participate actively in mathematical conversations, since if they were not active it would be hard for them to develop their communication and reasoning abilities. In the first cycle, the findings about student interactions and perceptions provide a base for how tasks and strategies could be developed in the following cycles in order to promote student-to-student interaction.

The analysis and exploration phase

In this phase, the context of the class was analysed by observing the mathematics lessons for a month. The observations indicated that almost all lessons had the same structure: a whole-class discussion about the content in a movie that the students had watched as homework and after that the students worked with textbook tasks while they were seated in groups of four students. Sometimes the teacher gave them a group task.

In some groups, there was very little mathematical communication with only some students being active. Often, the students continued to work individually or some students dominated the conversations. Fuentes's (2013) research had noted similar problems in the group work she observed.

The design phase

Consequently, tasks for promoting student-to-student interaction as well as means of support for helping students to communicate were designed in cooperation with the teacher. The students were divided into new groups, still with four students in each group. In previous research (see Deen & Zuidema, 2008; Fuentes, 2013) groups of four had also been used.

As group interaction needed to be more effective, the focus for the tasks in this first cycle became not to introduce new mathematical concepts, but to increase the quantity and quality of student interactions, based on the IC-model (Alrø & Skovsmose, 2004). The tasks were designed to support students to use the dialogic acts of the model by talking to each other (*getting in contact*), understanding the problem (*locating*) and trying out different problem-solving strategies (*advocating*). It was considered that their conversations could include the acts of *identifying, thinking aloud* and *reformulating*, depending on the content of the conversations. The act *evaluating* would be covered in the final whole class discussion, but could occur also in the group talks. At this stage, it was decided not to focus on supporting students to *challenge* each other's ideas. Instead the teacher was to do this when visiting the groups. This was one of the differences from using the IC-model to understand teacher-student interaction, which the model initially was developed for, to focusing on student-to-student interaction. A teacher's role can include naturally *challenging* of students' mathematical thinking, but students may not consider that they should follow up on others' utterances and ask for clarifications or justifications of claims.

The first problem that the students had to solve in groups involved fractions:

> Marie and Johannes need to paint a fence. If Marie does the painting herself it will take 4 hours. If Johannes does it, it will only take 2 hours, since he has a broader brush. They need 10 litres of paint for the fence. How long will it take to paint the fence if they cooperate and paint the fence together?

The second problem described a competition between groups of students that could best be solved with the help of probability reasoning. The problem was:

> Two dice are thrown. Guess the sum of the dots on the dice to win the game.

At the end of the lesson, each group had to pic one number between 1 and 12, and the group with the right guess won chocolate when the dice were thrown. The task was to reason mathematically together about which number to pic in order to increase the chance to win the game.

The analysis/exploration phase had reinforced Sfard and Kieran's (2001) warning that "the art of communicating has to be taught" (p. 71). So to do this, it was decided to give the students: a sheet about problem solving, a question list, and roles in the group work. There were several reasons for choosing these means of support. Rojas-Drummond and Mercer (2003) claimed that it is important to teach the procedures for problem solving. Hence, students were given a list of questions for when they began a problem-solving task. Mercer (1995) claimed that when teachers ask students questions, students get at chance to "check, refine and elaborate" (p. 10) and in this cycle it was considered that students could help each other to do this with the support of a question list and by writing down the group's important mathematical questions. These would be followed-up in a whole-class discussion at the end of the lesson. Finally, to support the students becoming positively interdependent on each other, the group roles identified different responsibilities. The roles were: Chairperson, who was responsible for deciding who talked when; Summarizer, who was responsible for making short summaries about what the group concluded; Thinker, who was responsible for talking aloud about his/her thoughts; and Accountant, who was responsible for showing the group's solutions to the teacher and/or the class. All of the students were Questioners and so expected to ask each other questions.

Results from the design phase

To determine if the two groups' interactions had improved from what was seen in the initial observations, students' utterances were compared to different acts in the IC-model. Interviews and questionnaire responses were used to verify classifying the utterances according to this model. As the evaluation of the first cycle, this material provides base line data for comparisons with later cycles. This comparison will contribute to responding to the two research questions, particularly the one about changes to interactions over time.

One group consisted of four boys, who all spoke different first languages. One boy, Carlos, started in the class a few weeks later than the others. Usually during the lessons, they were loud but on task. Azad had a leading position in the group. He talked often and enjoyed explaining mathematics to the others.

During the task about the fence, Azad had the role of the Accountant but talked most of the time. Meanwhile, Carlos, who was the Thinker, only expressed his opinion a few times during the twenty-minute conversation. Another boy, Mustafa, who was the Summarizer, was quiet in the beginning, but after some thinking-time started communicating with the others. Mohammed, the Chairperson, was active throughout the discussion, but did not take on the role as Chairperson. Instead he talked to his peers as he usually did.

All four students were focused on the task about the fence and initially there seemed to be a lot of *getting in contact* and *locating* when the students tried to understand how the painting of the wall could be divided between the persons and how much time different fractions of the wall would take for them to paint. At the same time, there was a kind of competition about who should be speaking, especially between Mustafa and Azad. They were not always competitive and often ended their sentences with tag questions, such as "okay?" or "do you understand?". This can be connected to the act *getting in contact*, which Alrø and

Skovsmose (2004) described as “tuning in on the co-participant and his or her perspectives” (p. 101). The tag questions made it possible to ensure that the other students could follow the mathematical reasoning.

For the task about the dice sum, the roles were changed and Carlos took a more dominant role as Chairperson. He was active in the discussions and everyone got more space to talk except Azad, who was grumpish and frustrated that he could not talk as much as he used to. The competition about talking time continued. However, in the first cycle interviews, the four boys stated that they liked working together. In the questionnaire, there were no clear differences related to how much they talked. Carlos, who talked the least, claimed that he was active in the discussions and that they all listened to each other. He thought that working with different roles was good. The only one who thought the roles did not work was Azad, who said that everyone just talked the way they wanted.

In the interview Mohammed said that Azad talked a lot, but that this was good. He called Azad “the king” and said that he liked that someone was the leader in the discussions, since otherwise it was hard to know what to do. However, although Mohammed focused on the benefits of this, Mercer (1995) warned that when students have different mathematical knowledge, it may be that a student “who dominates decision-making and insists on the use of their own problem-solving strategies may hinder rather then help the less able” (p. 93).

The group did not have time to finish the task about the fence. When the solutions were presented in a whole-class discussion, Azad said “The task was easy, but we made it much harder than it was. I actually felt stupid after I saw the answer”. (Den var enkel, fast vi gjorde den mycket svårare än den var. Jag kände mig dum efter jag fick se svaret faktiskt).

In another group, two of the group members were the girls, Aisha and Mariam. They worked closely together for both tasks, while the other group members varied. There was a lot of *reformulating* as they continuously completed each other’s sentences and helped each other with calculations during the conversation. From the recordings it was not possible to determine who had which role, which suggests that they did not follow the roles. When the group could not find the right answer, they became frustrated. They focused on the word “motivera” (justify) in the task. Another girl in the group, Nour, stated:

Nour: I hate when they say justify. I hate that word, in all school subjects. Yes. Justify. What do they mean justify? Especially in maths. You cannot justify. You think. Justify. It is something inside your head. (Jag hatar när de säger motivera. Jag hatar det här ordet, i alla ämnen. Ja. Motivera. Vadå motivera? Särskilt i matte. Man kan inte motivera. Man tänker. Motivera. Det är alltså något man har i huvudet.)

The girls tried to *get in contact* and *locate* the mathematics in the problem, but did not succeed as they started with guessing an answer and then trying to count backwards, not using fractions that they currently were working with during mathematics lessons. Alrø and Skovsmose (2004) claimed that emotive aspects, such as mutual respect, responsibility and confidence are important for the learning process and that there might be a risk that “the loss of contact became a hindrance for the co-operation” (p. 101). The group got stuck because they could not find the correct answer and they did not know how to mathematically justify their guesses. The general advice about using the problem-solving sheet did not help and they were not *challenged* in their thinking.

During the task about the dice sum, Mariam and Aisha’s group continued to focus on getting the correct answer. Such an approach has been identified as problematic. Mercer (1995) stated that “students may be more worried about ‘doing the right thing’ than with thinking things through” (p. 28). Another issue for this group was that there was a lot of focus on students’ attitudes to mathematics, such as the discussion about justifications. Another example is when Aisha and Mariam, talking over the top of each other, claimed:

Aisha/Mariam: But how? We cannot win, they are better… but you have to try. We are not… We are so stupid compared to the others. We are. We are. (Alltså hur? Vi kommer inte ens vinna, de är bättre… alltså du måste försöka göra det. Vi är inte… Vi är så dumma jämfört med de andra. Det är vi. Det är vi.)

In the interviews the girls claimed that much of their feelings about being stupid could be because they did not find the correct solution. They claimed that it was central to try out different problem-solving strategies, but that they were very focused on the answers. However, when Aisha stated that she was no good at mathematics in the interview, Mariam and another group member told her that it was untrue and reminded Aisha that she had helped them with mathematics tasks earlier that day. The atmosphere in the group seemed supportive.

The evaluation phase and implications for the second design cycle

The analysis of the group work contributed to the evaluation phase in which design ideas and tasks are empirically tested (McKenney & Reeves, 2012). In the evaluation phase conclusions are made about which pedagogical aspects need to be reconsidered in the next cycle and how tasks and means of support need to be changed to promote student-to-student communication. The results implied that Walshaw and Anthony’s (2008) thoughts that group work promotes social and cognitive engagement were only partly shown in the first cycle. Although the students did actively engage and talk about the mathematical content and worked with problem-solving strategies, their contributions varied. Limited attempts to follow the roles were made. They did not use their question lists actively, which made a meta-level whole-class discussion about questions difficult.

Consequently in the second cycle, the plan is to refine the strategies to support students’ interaction. Alrø and Skovsmose (2004) mentioned that finding a fully developed IC-model is rare, and the results of the analysis showed that only certain elements of the IC-model were identifiable in this first cycle.

There were also unexpected findings such as students' feelings about being stupid or competition to dominate the conversations, which needed to be dealt with in the next cycle if the interactions were to improve students' opportunities to learn mathematics. As Esmonde (2009) claimed, group work can produce "undesirable social interaction styles" (p. 1009). Another problem was that the groups were very focused on getting the correct answer and not on using different problem-solving strategies. Therefore, in the second cycle, tasks will be chosen that have more than one answer. Also, students will be encouraged to ask quiet group members questions and to try different strategies to solve problems.

Another result from this cycle was that most of the students did not follow the roles, so changes are needed in the descriptions of the roles. Esmonde (2009) claimed that group roles contribute to equitable learning opportunities, only if students consider the roles to be important, understand the reasons for them and agree to try them out. For instance, since no one listened to the Chairperson about who could talk, there is no reason to include this role for the new tasks. In one of the groups, Azad took the role of the leader, without having this as his designated role, yet the others seemed to accept this. For the new tasks, there will be a Groupwork-leader responsible for thinking about the group and if someone is too quiet to ask him/her questions. There will also be a Questioner responsible for highlighting mathematical questions, at least one from each person in the group, a Writer responsible for the written report to the teacher and a Teller responsible for telling the rest of the class about the solutions. After the task there will be a meta-discussion about the roles and how the cooperation worked and a second attempt at a meta-level discussion on mathematical questions.

Another factor that may affect how the groups worked is different students' needs. For instance on the task about the fence, Mustafa claimed in the interview that he needed some time to think about the task before he entered the discussion, while Azad started talking straight away. For the next cycle, some individual thinking time will be added before the group discussions begin so that everyone gets a chance to prepare for making a contribution.

Conclusion

The aim of this EDR-study is to improve understandings about how to increase student-to-student interactions both from the task design perspective but also from the students' own perspective. This was deemed as important both because of the new emphases in the syllabus but also because initial observations showed limited mathematical communication in relation to the acts in the IC-model occurring in the classroom. Results from the first cycle show that it was possible to improve students' contributions to mathematical discussions about problem-solving tasks. However, some strategies needed to be changed for the next cycle so that the quality as well as the quantity of students' contributions increases.

These changes include designing tasks in order to avoid the search for the right answer and using strategies that make students more confident about their mathematical abilities, for instance through making the roles more interactive in that the students invite each other to contribute to the group discussions. The social structures in the groups and students' attitudes towards mathematics are shown to be important.

The strength of using an EDR-approach in this project is that the cyclic nature makes it possible to improve the designs in a flexible way to meet the needs of the students, needs that may not be apparent before the first task. For example, the first cycle indicated that students' perceptions of how good they are at mathematics and their attitudes to problem-solving situations are important features and need to be taken into consideration when trying to promote rich learning opportunities.

EDR is also a method for researchers in mathematics education to find and improve theoretical tools for studying student communication and develop deeper understanding on student-to-student interaction. In this project, the first cycle shows that there is a need to analyse the structure of the student interaction in more depth. The acts in the IC-model, although useful for planning activities, seemed to not be so helpful when analysing data. For the second cycle, the theoretical base in the design phase will be complemented by Fuentes's (2013) framework for analysing student communication. This framework contains eight different communication patterns between students, which will be used with an analysis of the different dialogic acts from the IC-model that appeared in the students' interactions.

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Spaces of Values: What is Available to be Adopted by Students?

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We consider the space of values *to which students are exposed through teacher utterances in relation to the tasks provided and the nature of the interactions between with students. We restrict attention to values associated with care for mathematics and care for students because there are endemic tensions both in and between these. We are concerned here only with what is available for students to detect and internalise, not what is actually internalised. We illustrate the notion of a space of values by considering the behaviour of one teacher, selected from a group of previously studied teachers (Skilling, 2013), all of whom were recognised for maintaining a high degree of engagement of their students.*

Theoretical Frame

Teaching mathematics can usefully be seen as a caring profession (Trigwell & Prosser, 1996; Goldstein, 2002; Mason, 2002; Noddings, 2012; DeVito, 2006). As in any caring profession it is vital for the effectiveness of their actions that practitioners display both *care for* the people they serve and *care in* the exercise of their profession. However, there are endemic tensions: on the one hand, there is a tension between caring for the student and being seen by students to be caring for them, and on the other hand, there is a tension between caring for students and caring for mathematics. For example, wanting students to 'have fun' is an extreme form of caring for students, and while positive affect is important, 'caring only for students' can all too easily displace providing contact with significant mathematical thinking (Heaton, 1992; Moyer, 2002), thus losing contact with caring for mathematics (and students' mathematical development); concentrating on mathematical reasoning can all too easily leave students bewildered and frustrated (caring for mathematics at the expense of caring for students). When students struggle with a task, teachers can be tempted to simplify the task so that it can be accomplished (Stein, Grover & Henningsen, 1996), which displays care for students at the expense of mathematics. Our interest is in tensions between care for both students and for mathematics, and how this care is available to students as a space of manifested values.

Care is shown through what is valued by the teacher and the institution, hence our interest in what might constitute the *space of values* accessible to students through immersion in the milieu and ethos, and through interactions with both teachers and mathematics.

In his landmark book, Bishop (1988, pp. 60-81) identified three clusters of values often identified with mathematics, each with two aspects: ideology (rationalism and objectism), sentiment (control and progress) and sociology (openness and mystery). Our interest is not so much in these very general values, but rather in how more specific values are made available, transacted and even promoted through student-teacher and student-student interactions.

Bishop, Seah & Chin (2003) lay out the case for values as the focus of mathematics education curricula concerns. For example, they make the point, and refer to many other authors making the same point, that "there is widespread agreement among writers about values in education that whenever and wherever any teaching takes place, values are being taught and learned (p. 718, see also p. 721)". But the word 'values' is used variously to refer to ethical, moral, political, philosophical and spiritual dimensions, as well as to social, cognitive and psychological experiences. Here we restrict our attention to the domain of mathematics: encountering and experiencing mathematical thinking in classrooms. The *values* of interest are the apparent values of the teacher as to how mathematics is learned and done, and how students are supported, or in other words, how the teacher displays care for mathematics and care for students. Teacher displays of care are reflected by interactions with students and the mathematical content within the context of each classroom environment. This is our starting point because this is how care is manifested.

Our interest, initially, is in what values are available to be interpreted as such by students. Thus we distinguish between values espoused in private, espoused with students, and available to be experienced by students, focusing principally on the latter, though using the former two as a guide. Of course we acknowledge that we are not privy to what attention students pay to the values that the teachers espouse and display and can only interpret this from the actions and interactions observed in the classroom.

We take our lead from *variation theory* which highlights the *space of learning* associated with tasks and interactions, focusing on 'what is available to be learned' because the student has experienced variation in its critical dimensions (Marton & Booth, 1997). Values require a different theory however, because although it seems clear that students are unlikely to pick up unmanifested values, it is not clear how a space of values is opened up for students. It is certainly not clear that variation in enacted values, or even variation in how the same value is enacted, are necessarily relevant. Studies such as Perry (1968), and Copes (1982) who tried to use Perry positions in mathematics teaching, indicate the complexity of the issue. This paper is an initial foray into this domain, using previously collected video-transcripts of teachers with a record of engaging students in mathematics as data for testing whether the construct of a *space of values* makes sense.

We see values concerning how mathematics is approached and engaged in as being experienced by the full psyche: behaviour-enaction, emotions-affect, intellect-cognition, and attention-will (Mason, 2003) via the construction or adaptation of one or more 'mathematical selves' which channel energies in characteristic ways. Thus we aim to probe beneath the surface of socio-mathematical norms (Yackel & Cobb, 1996) which concentrate on practices, to consider what values are manifested. We are interested in features such as sense-of-coherence, appropriate challenge (Jaworski, 1994), respect and trust so that significant mathematical and personal choices are possible, the kind of support provided during periods of frustration and not-knowing, as well as recognition of the frustrations when coming-to-know. There are obvious connections with self-efficacy, agency and many other socio-psychological constructs too numerous to mention much less integrate into this paper. We anticipate that during interactions in mathematics classrooms values will at times appear to stress care for students, for mathematics or possibly both.

Gathering Evidence

A recent case study investigated the beliefs and practices of four teachers who were identified as promoting and maintaining engagement in mathematics classrooms (Skilling, 2013). The rationale for investigating different cases was to identify both shared and distinctive beliefs and practices amongst teachers of students with high and low levels of achievement.

The study drew upon multiple sources of data such as teacher surveys, pre- and post-lesson interviews with the teachers, and lesson observations. In this way teachers' self-reported beliefs could be compared to their observed practices. Student engagement is conjectured here to be shaped and influenced by values displayed in and through various teacher practices, including dimensions such as the nature and quality of student-teacher interactions (Skinner & Pitzer, 2012), individual teacher differences (Hardré, Davis, & Sullivan, 2008) and levels of teacher support for students (Midgley, Feldlaufer, & Eccles, 1989). The teachers all expressed belief that student engagement was an important element for learning mathematics (e.g., through responses to a Teacher Beliefs and Practices Survey (Skilling, 2013)) which was reflected by their use of supportive positive motivational factors in approaches to lesson planning and responses to students' needs (Anthony & Walshaw, 2007; Doig, 2005; Sullivan, 2011).

As with the construct *space of learning*, there is no claim that students were influenced by the values displayed. Our aim is to find a way to describe what is available to be experienced, so that later studies can explore what values are taken up by students, under what conditions, and in what ways.

Methods

We considered data from several teachers but because of limited space, selected a short segment from one teacher's classes which seemed to us to highlight most clearly a range of values. In a longer paper we could have offered data from several teachers and chosen longer and different sequences, and these might perhaps have shown further variation. Our concern here is with the notion of space of values, not a comprehensively phenomenographic study of such spaces. The video-transcript was trimmed down to teacher utterances while viewings of video recordings generated detailed descriptions of events and actions, although we are well aware that there were other things going on at the same time which could impinge on values. We then considered the range and the degree of repetition of various utterances and actions and interpreted these as displaying values associated with care for mathematics, care for students, or both. Because this is only an initial enquiry, it is not appropriate to undertake any form of triangulation or testing of inter-rater reliability when analysing transcripts, although the initial study (Skilling, 2013) incorporated these.

Mr. Tower

For this paper an excerpt from one observed lesson (lasting from the 6th to 13th minute of the 50 minute lesson) has been selected in order to explore details of how Mr Tower's behaviours, utterances and interactions with his students, revealed his care for both students and mathematics and how these values were displayed during the lesson. The lesson was on the topic of mass, and included discussions on what was meant by mass, units used commonly (in Australia) to measure mass, relationships between units, converting from one unit to another, and assigning units when weighing objects. The class was one of two higher achieving classes in Year 7 as assessed by the school at time of entry to secondary school, and Mr Tower reported that he was mindful of maintaining a good pace of learning to meet what he considered to be the learning needs of these particular students. The excerpt comprises all his utterances over the period (see Appendix A) which involved identifying, ordering, abbreviating and converting units for measuring mass.

For the present study, the authors were interested in examining what seemed to be valued by Mr Tower concerning how mathematics is approached and engaged with, as he went about interacting with the students in his class. His instructional style included phases of asking students to record their thoughts on mini-whiteboards, discussing individual responses with other students, clarifying concepts for and with the class, and providing time for individual students to reflect on and make adjustments to their understanding of concepts. He usually walked around the room reiterating the task request, praising student efforts, affirming student progress and attending to individual students who he assessed as requiring support.

Overall analysis

Several overarching values were interpreted as being displayed. First, a wide range and variety of largely consistent values were displayed throughout the chosen lesson segment, indicating something of the depth and complexity of events and interactions occurring in learning environments. Second, the timing and frequency of particular values was a particularly notable feature. Some of the values displayed were interpreted by us as predominantly orientated towards displaying caring about mathematics, others were interpreted as being predominantly orientated toward displaying caring for and about the students, while others combined both, or could be taken either way. The range and extent of these suggest that, being multiply construable, how they influence students' adoption of corresponding values is likely to be complicated. Although recognising that the same act can be interpreted as displaying a range of different values, the following discussion aims to describe what and how the values portrayed were meaningful in terms of learning mathematics and for students as learners of mathematics. T-codes refer to the data which is in the appendix.

While any action initiated by a teacher can be considered to exhibit concern if not care for students' mathematical wellbeing, and justified as such, we distinguish between actions for which the focus is predominantly the correctness, structure and meaning of the mathematics and actions for which the focus is predominantly the students state, which includes cognitive, affective, enactive and attention-focus.

Values particularly associated with caring about mathematics

In this category we place actions and utterances that we interpret as being focused on clarification, emphasis on conventions, on students' utterances being mathematically correct or appropriate, on feedback apparently aimed at exposing everyone to correctness, and on making or promoting connections with prior concepts.

In the excerpt, values of class and individual construal and meaning making were often combined. For example, when Mr Tower asked students to record words used to describe units of mass, to order units of mass, and to recall abbreviations for units of mass (T1, T7, T12) he appeared to value not only what individual students construed but the extent to which the whole class made sense of the concepts. This aligns with the notion Davis (2005) put forward of the teacher as the 'consciousness of the collective'.

We interpreted mathematical organisation, connections and conventions as being valued throughout the lesson. For example, when Mr Tower asked students to "order" units of measurement "lightest first" (T6), and by asking students "what is the connection between" (T12) and "how would you show how you might change or convert" (T17), the students were challenged to demonstrate and connect what they knew from prior learning experiences. Mathematical conventions were valued when abbreviations for different units were clarified (T11) and the connection between units established (T12). Mr Tower sought to clarify terminology and processes by asking students to re-state concepts (T5) or by re-stating concepts himself (T5, T11, T14), emphasising making meaning of converting units of measurement.

Many of these values would only emerge as values if they were to occur repeatedly, or if they were treated to scaffolding and fading (Seeley Brown, Duguid & Collins 1989) by making prompts increasingly indirect so that students begin to internalise them for themselves (Love & Mason 1992). On the two occasions that Mr Tower was observed teaching, consistency with the values he portrayed about caring for mathematics was evidenced by the emphasis he placed on being clear and precise, and on connecting mathematical ideas. The way that students responded throughout the lessons to the questions and tasks asked by Mr Tower indicated that the value of caring about mathematics was adopted as a 'normal' expectation in this classroom.

Values particularly associated with caring about students as learners of mathematics

Practices which we interpret as indicating caring for students include: providing regular class and individual affirmation and praise; opportunities for reflection and self-regulation; alleviation of anxiety; acknowledgement of persistence; maintaining interest; acknowledging task difficulty; and opportunities for collaboration. Many of these practices that support student learning are discussed as affective factors in terms of motivation engagement research, and certainly attending to students emotional needs is crucial for influencing learning outcomes (Hannula, 2004).

Mr Tower demonstrated his interest and valuing in assessing student progress on numerous occasions stating that "I am going to get you to show me" (T1), and asking student to hold up their work so that he could gauge student thinking (T3, T7, T9 and T20). This could be interpreted as caring for the mathematics over caring for students where students might feel embarrassment or negativity about being asked to expose their thinking to others, but we did not detect any such reluctance or negativity: everyone asked seemed content to expose their thinking. This is in alignment with the notion of a conjecturing atmosphere (Mason, 2003).

Associated with assessment, both feedback and clarification were provided during the lesson, which can be interpreted as concern for students to understand fully, but could be interpreted as testing students' understanding. Upon observing that some students had identified four units of measurement and others three, rather than correct individuals, he stated that "Some have written more than others" (T4) and asked the class to share and consolidate the four units of measurement that they would be expected to use.

Value judgements in the form of affirmation of students' progress were observed as being directed both toward the whole class as well as towards individual students. For example, comments such as "Most people are doing that very well" (T3) were directed to the class, whereas "Good, all correct" (T14) was directed to the student who was asked to complete work on the board at the front of the classroom. It was also observed that Mr Tower repeatedly affirmed student progress with combined general comments such as: "Right. Good. Okay" (T3, T4, T6, T9, T13, T18 and T21). However, on a number of occasions these affirmations were more explicit such as: "Everyone is on target' (T3); "Most of you have got this" (T13); "Some of you have got the idea"; and "I like what I see guys. This is very good" (T20). Whether these utterances become a weakened currency due to their frequency would be a matter of further study by listening to what students have to say. His constant movement around the classroom looking at individual's work could be interpreted as coercive, or as displaying care for students. However, it was observed that students did not hide work and in many instance offered their work for Mr Tower to see, which suggests that the students valued and felt comfortable with having their mathematics work checked over.

Mr. Tower took several opportunities to show that he valued students reflecting on their learning. For example, students were encouraged to "have a look around and see other people's [work]" (T20) and make adjustments to their work by collaborating with their classmates (T10, T13 and T20). Additionally, student self-regulation was encouraged ("You might have to change your whiteboards now if you didn't quite get that", T15) indicating that student autonomy and clarity of concepts were valued. Again, the practice of viewing the work of others emphasised the importance that Mr Tower placed on 'collective' understanding of concepts by everyone in the class. We could infer that Mr Tower's values for caring about mathematics underpin his caring about individual understanding and he extends this notion of caring to include all the members of the class. In this way not only are his values for caring about mathematics displayed but Mr Tower also models that each student should care about their own and others mathematical understanding.

Alleviating possible student anxiety showed care for students' affect: "If you're not sure, you're not sure. That's okay" (T8) and checking for full student clarity was indicated by "Did everyone hear that? Who wants it said again?" (T5). Mr Tower also displayed valuing challenge: "Okay, these are hard questions. This is going to challenge some of you" (T16). Coupled with challenges Mr Tower also supported students by acknowledging their efforts ("I know you will do your best", T2) and persistence, as well as the need for variety: "We are going to do a little bit more with me and then we are going to get you guys to do some" (T19). Apart from affirmation throughout this part of the lesson, Mr Tower's final remark of praise was linked to values of expectation and satisfaction signalling that the class were ready to move on to the next phase of the lesson: "Alright, that's brilliant. Well done guys. So, we are ready to convert" (T21).

Tentative Conclusions

The same action by a teacher could be interpreted positively, neutrally, or negatively by students. For example, being asked to expose your working to the whole class can be seen as positive in a conjecturing atmosphere focused on learning and developing, as neutral when simply accepted as a classroom practice and as negative when emphasis is on correctness and competition. Indeed different students may interpret the same act differently. What really matters is how the classroom ethos is developed and practices introduced, including the stance taken by the teacher through what is said and done to indicate what the teacher values.

We are struck by the complexity of the range and intensity of values that are available to be interpreted by students from immersion in different classroom practices and milieu. Any *space of values* being displayed and made available to be adopted by students can be nullified through inconsistency, can be neutralised through becoming an un-reflected-upon practice, or can be exposed through repetition, meta-questioning, and through the tenor of the relationships that the teacher has with the students and with mathematics, and how these play out together.

We suspect that it could be useful to teachers and teacher educators to become aware of unintended values being interpreted by students from habitual classroom behaviours (ways of working, ways of speaking) which are not in alignment with espoused values, especially in connection with care for mathematics, and care for students. This could inform pedagogical choices involving both the selection of tasks, the sequencing of content, asking questions, and ways of promoting interacting with students and with mathematics, including being mathematical with and in front of students.

Our initial analysis also indicates that trying to assign specific values to specific actions will be less fruitful than maintaining the complexity of human interactions. Sometimes care for students and for mathematics are in tension, and sometimes they are in harmony.

What we have learnt from this initial foray into considering values for caring about mathematics and caring about students as learners of mathematics is that it is not so much the utterances themselves, nor even the actions of which they form a part, but the entire ethos of the classroom that is likely to influence how students respond to the care being displayed, how they interpret the values being enacted, and whether these are taken up as values in the long term or acceded to as practices in the short term. It is also likely that students go through periods of frustration as well as elation, and it is how these energies are perceived and handled that is likely to influence students. Extending this research to investigate student perceptions of what they value in mathematics classrooms would complement the findings and broaden our understanding of how the *space of values* can hinder or promote student learning in mathematics.

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Appendix: Mr. Towers' utterances

T1	"I want you to write down the words we use when we measure mass" REPEATS "All the words you can think of". "Write them down and then I am going to get you to show me"
T2	"Don't worry about spelling, that's okay. I know you will do your best"
T3	"Good, most people are doing that very well" ; "Okay, hold them up so I can see them" ; "Good. Right. Okay. Everyone is on target"
T4	"Some have written more than others. Hands up those who wrote three names down? Who wrote four? Good."

T5	"What are the four?" Student A: "Grams, kilograms, tonnes and milligrams". "Did everyone hear that? Who wants it said again"
T6	"I want you to write them in order, those four words, lightest first – lightest first". "Good. Okay. That's good. Okay"
T7	"Now before I get you to hold them up, I want to see if you know their abbreviations. Write the letters that abbreviate them next to the words"
T8	"If you're not sure, you're not sure, that's okay"
T9	"Let's see, let's go—hold them up! Very good, very good."
T10	"Check it with the person next to you as well. Have a look at theirs".
T11	"Right, so you should have had: milligrams in brackets mg, grams in brackets g, kilograms in brackets kg, and tonnes in brackets t. How do you say that word? Some say t**o**nnes but we will say tunnes—we are used to that"
T12	"Now we are going to see what the connection is. I want to see how much you know about this. What is the connection between grams and milligrams?"
T13	"Good. Most of you have got this" REPEATS (Checking individuals' work) "Check with the person next to you"
T14	"Sarah, go out and write the answers on the board at the front for me. Good all correct. One thousand, one thousand and a thousand. Thank you" "Is she right?" Class responds "Yes"
T15	"Very good. So, you might have to change your whiteboards now if you didn't quite get that. So they were all a thousand"
T16	Stands at the front, pauses and holds out hand to gain students focus. "Okay, these are hard questions. This is going to challenge some of you".
T17	I want you to come up with a creative way of how you would show how you might change or 'convert' (points to this word on the board) grams to milligrams, kilograms to grams and tonnes to kilograms". Re-phrases: "How might you combine all of that information to say I am going from tonnes into kilograms, kilograms into grams and then the reverse?" "Now you have seen this before and you could repeat it but you might be able to come up with your own way. How could you connect?"
T18	"Okay, some of you have got the idea. Very good, very good. A lot of you have remembered past ways of doing it"
T19	"Okay, I have been out here working for a long time. We are going to do a little bit more with me and then we are going to get you guys to do some"
T20	"Right, show it to the person next to you. I like what I see guys. This is very good. If someone next to you has got it wrong or they have made a little mistake…give them a little bit of support…now hold up…have a look around and see other peoples as well"
T21	"Alright, that's brilliant. Well done guys. So, we are ready to convert"

Learning Subtraction Strategies From Principle-Based Teaching Activities

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Three teachers and a researcher have co-designed a teaching activity intended to support students' learning of two strategies for subtraction. The researcher focuses on the relation between theoretical principles, introduced to underpin the participating teachers' work, and the learning outcomes of their 33 students in grade 4. The principles are adapted by the researcher during three design cycles and negotiated with the teachers to meet emerging needs in the design process. The three teachers are fully responsible for planning, implementing, and evaluating an iterated teaching activity designed according to these principles. This study indicates positive effects of targeting low-achievers with teacher-led structured group activities, using guiding principles from self-regulation theory.

Introduction

Researching the teaching and learning of mathematics usually involves providing theoretically grounded descriptions of observed classroom activities or learning processes. This is often done without intervening in these activities and processes. In contrast, design research explicitly addresses the provision of opportunities for learning (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003). Design researchers engage in a cyclic process involving both instructional design and classroom-based research, encompassing all aspects of a teachers' work with planning, implementing, and evaluating teaching activities (Cobb, Stephan, McClain, and Gravemeijer, 2001). Such a comprehensive approach involves several different research tasks, implying high demands on the design researchers' cognitive, material, and social resources (Boote, 2010).

In the study reported in this paper, we have adopted a more modest and less demanding approach to design research. As researchers, we do not observe the classroom activities and we do not engage in qualitative analysis of the learning outcomes. Instead, we focus our attention on underpinning theoretical principles that may improve the learning outcomes. The researcher introduces and adapts theoretical principles to emerging needs in the design process, while the participating teachers are fully responsible for planning, implementing and evaluating an iterated teaching activity designed according to these principles.

Our study may be compared with a (preliminary) clinical trial, where specific treatments are introduced in an ecological context. Clinical trials are commonly used in medicine, for example to identify effects of various drug treatments. If a preliminary study indicates positive effects, repeated studies may be carried out for the purpose of confirming these effects. Although such positive effects may be confirmed, they are seldom explained. We follow a similar rationale in our research, that is, we attempt to identify positive effects (as improved test scores) of “theoretical treatments” adapted to meet emerging needs in the design process. Although the current study puts focus on identifying specific principles for treating a specific issue, it also investigates the potential value of using the methodology of principle-based clinical trials in mathematics education.

Research objectives

Our objective is to investigate possible connections between underpinning principles for teaching activities and students’ test scores in relation to the specific learning object of these activities. As principles, we consider theories and theory-based methods that are introduced by the researcher and guide the teachers’ planning and implementation of teaching activities. The principles are updated in a cyclic process, based on the students’ intermediate test scores and the teachers’ observations. Our research question follows.

- How does the flexible outcome-based adaptation of underpinning principles for the teaching activity affect the students’ test scores?

The study involved three teachers and 33 students in grade 4 in Sweden. The learning object concerned contrasting, selecting, and applying two different strategies for subtraction, namely adding up (as in 304 – 298 = 2 + 4 = 6) and subtracting parts (as in 435 – 121 = 300 + 10 + 4 = 314).

Our conceptual framework – a bricolage of theories

Our study applies a *bricolage* of theories of different character and from different research traditions (Kincheloe, 2001). The bricolage approach, which fits within the Singerian inquiry tradition (Lester, 2005), has a long tradition in mathematics education research and challenges “the positivist epistemology of practice wherein practical reason is construed as the application of theory” (Cobb, 2007, p. 3). While a Lockean inquiry regards observations as evidence with respect to pre-defined theories, the Singerian inquiry “entails a constant questioning of the assumptions” (Lester, 2005, p. 463). Instead of generating research questions that fit a specific theoretical framework, our bricolage of theories is adapted to authentic questions and needs as expressed by the participating teachers. Our bricolage is also adapted to the specific learning object, which necessarily involves representing the two strategies for subtraction. In the next section, we briefly discuss theories about representation of mathematical objects. In the last section, we discuss meta-cognitive strategies for self-regulation. While theories of representation were included as principles from the beginning of the project, the theory of self-regulation became involved in the third cycle. In addition to describing these theories, we briefly account for how they were introduced – *but not how they were used* – as underpinning principles for the teaching activities.

Mathematical representations

At the first project meeting in February 2012, subtraction was discussed from a structural perspective as a mathematical idea that needs to be mediated (or represented) by the use of artefacts (Ogden & Richards, 1923; Duval, 2006; Winsløw, 2003) such as tangibles, pictures, diagrams, symbols, and natural language. Representations can be transformed in two qualitatively different ways (Duval, 2006): as *treatment* within a specific representational system (e.g. the symbolic treatment 34 + 25 = 50 + 9) and *conversion* between different systems (e.g. converting three apples to the symbol 3). Ability to make conversions and coordinate different representations of the same object is needed for conceptual development (Winsløw, 2003) as well as problem solving (Janvier, 1987).

The participating teachers were well aware that the two targeted strategies for subtraction could be represented in a variety of ways. Examples were shared about possible ways to represent the two strategies, for example by making use of the number line or tangibles such as measuring tape or pearls on a string. In addition, the researcher introduced the so-called empty number line, commonly used completely without markers (Klein, Beishuizen & Treffers, 1998), thereby inviting the students to add markers and numbers (Fig. 1a). To further stimulate the students to discover the adding up strategy it was decided to make use of number lines with markers but without numbers (Fig. 1b).

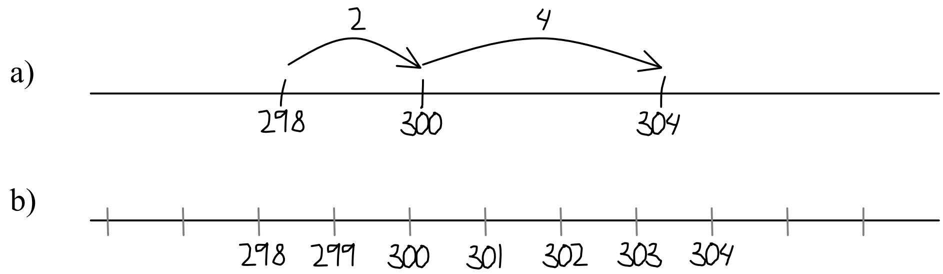

Figure 1. Two examples of empty number lines, completed by students.

Rather than asking the students to solve routine tasks by following instructions and making use of templates, it was agreed that the teachers should construct real-life problems inviting the students to work in small groups, exploring and modelling situations calling for them to compare or remove quantities by making use of provided artefacts. The teachers chose to avoid subtractions that do not fit well with respect to either strategy, for example 421 – 135. It was agreed to focus on the adding up strategy for terms that are close to each other (as in 304 – 298), and the by parts strategy when all the parts of the first term are larger than the corresponding parts of the second term (as in 435 – 121).

Self-regulation

At a team meeting during the third cycle of the design process, it was decided to draw on the theory of self-regulation. This decision was strongly influenced by the (rather disappointing) outcomes from the second cycle, where the poor test scores (for 13 out of 33 students) were interpreted as a consequence of the students not being able to distinguish between the two targeted strategies.

Mathematical problem solving or executing complex mathematical calculations often calls for ability to assess strategies and representations, select and implement a chosen strategy with a particular representation, monitor and control own performance of transformations, react on incorrect intermediate results, and reflect on the answer in relation to the original problem. Such meta-cognitive abilities are well aligned with the four phases of self-regulated learning: fore-thought, planning and activation; monitoring; control; reaction and reflection (Schunk, 2005). Each phase involves processes that can be related to cognition, motivation, behaviour, and context, or a combination thereof.

With respect to the two strategies for subtraction, the students are expected to assess that the subtraction 304 – 298 should be calculated by adding up, while 435 – 121 calls for subtraction by parts. Comparing strategies and thinking of different ways to represent these strategies (e.g. on a number line, or by splitting a number in its parts) is primarily a cognitive and contextual process in the phase forethought and planning. The students' mental representations may be externalized, but could just as well be managed internally (e.g. on a mental number line, or imagining a number being split in its parts). Still within this phase, the student has to select and activate a strategy with a chosen representation. This particular representation is often, but not always, externalized. The phases of monitoring and control require the student to engage in carrying out the strategy by transforming representations, either as treatments or conversions or combinations of both. Having achieved a preliminary answer, the student may reflect on its plausibility in relation to the original problem statement. For example, the student who transforms 304 – 298 to 194 could readily identify that the answer is incorrect by reflecting on the positions of 304 and 298 on a number line. Students who engage in systematic reflection during the transformation process may feel less need to engage in an overall reflection.

In our study, the teachers observed that not all students engaged in cognitive and contextual processes, particularly in the phase of forethought, planning, and activation. For this reason, the third design cycle specifically addressed such processes by targeting the low-achievers in structured teacher-led group activities.

Methodological considerations

The design process was documented by the researcher and one teacher, separately taking notes about progress and decisions. These notes were primarily used to keep the development project on track, but proved sufficient for supporting recall of associated events of relevance for the research study. The test scores were documented by the teachers during the project in tabular form.

The development of the students' test scores have been illustrated in line diagrams from where three different groups of students have been identified and characterized (Fig. 2). This simple approach has been manageable for our small sample of 33 students.

Regarding methods for organizing the design process, we have partly been committed to the current study being carried out within a development project arranged as collegial interaction with external expertise, as recommended by Timperley (2008). Three mathematics teachers have collaborated with one researcher in mathematics education. While Timperley (2008) addresses professional development of teachers, similar co-design approaches are well established in the research domain (Penuel, Roschelle & Shechtman, 2007). These approaches may be compared with theory-oriented design research as pursued by Cobb et al. (2001), with limited involvement of teachers, and the practice-oriented learning study approach (Marton and Pang, 2006) where teachers may collaborate without any guidance of external expertise.

In our principle-based approach, the teachers have been responsible for planning, implementing, and evaluating their own practices (Stigler & Hiebert, 1999; Clarke & Hollingsworth, 2002). The researcher has not engaged in qualitative analyses of implemented teaching activities and learning outcomes. The learning outcomes have been quantitatively measured on a traditional test prepared by the teachers, according to their own standards and not influenced by the principles. The researcher has only been responsible for introducing the theoretical principles and engaging in collaborative discussions with the teachers, with focus directed at motivating and exemplifying the principles, evaluating teaching outcomes and negotiating further actions based on these outcomes. The researcher participated in two preparatory meetings in spring 2012 and three additional meetings during autumn 2012, when the teaching activities were implemented with 33 students in grade 4. All of these meetings took place at the school in question. Between these meetings the teachers worked on their own to plan, implement and evaluate the teaching activities.

Results

The study was carried out in autumn 2012 with two classes in grade 4, each with 17 students. One student did not participate in any part of the study, and so the study consisted of 33 students. In the first and second cycles, the teaching activity had similar pedagogical arrangements. The two classes were taught separately, one hour per session, with most of the time spent on the students solving subtraction problems that called for making conversions between representations. The problem solving sessions were arranged with 3-4 students working together and the teacher walking between the groups to answer questions.

Intermediate outcomes influencing the design process

While evaluating the second implementation of the activity, the teachers discussed how to interpret the test scores. They agreed that 12 correctly answered problems – out of the total 17 problems – was a satisfactory result, but also noted that 13 of the 33 students had not achieved at least 12 points on either post-test 1 or post-test 2. On post-test 2, two students were close (11 points) but the remaining 11 scores were in the range from 2 to 8 points. These unsatisfactory results were discussed with the researcher. It was argued that the low achieving students may not have been involved in all aspects in the group work, possibly adopting passive roles and letting the other students dominate in the group work. The researcher and the teachers agreed to specifically target the 13 students and stimulate (force) all of them to become involved in the problem solving processes. It was agreed that the teacher should meet the (new) groups one at a time and spend 2x30 with each group, leading the work by asking questions to make sure all students become involved in all aspects of the problem solving process. An additional student attended although she had previously achieved a satisfactory result. For this reason, her test score on post-test 3 is not included in the diagrams. The 14 students were divided into four groups of 3, 3, 4, and 4 students, respectively. The 2x30 minutes per group were divided into 30 minutes each, on two consecutive days.

Development of the test scores

The test scores from post-test 3 show a substantial improvement for 9 of the 11 students, whose test scores increased from 2-8 points to 12-17 points (Fig. 2).

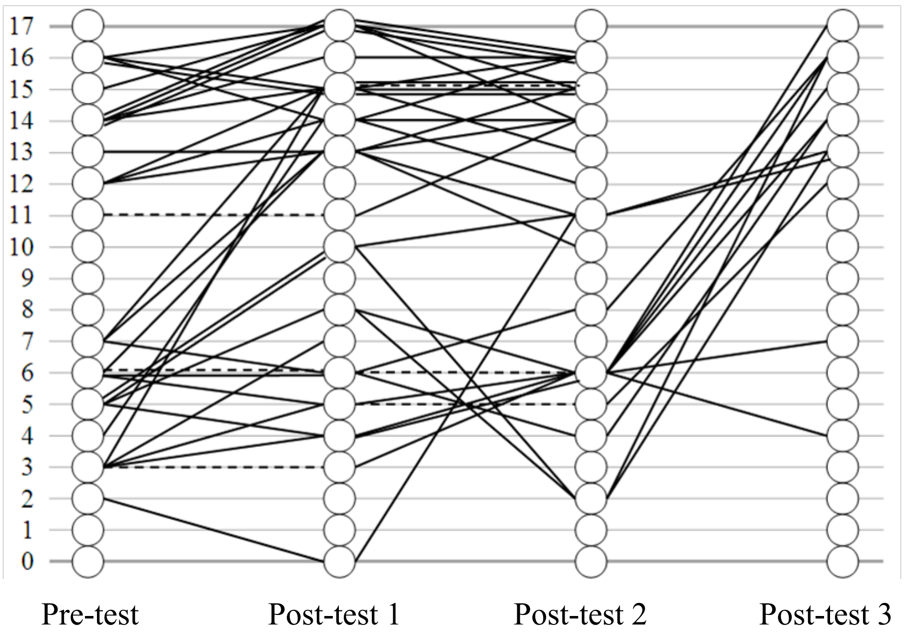

Figure 2. All test scores of the 33 students.

In addition, the two students who had 11 points on post-test 2 both achieved 13 points, thus also satisfactory. We would like to emphasize that the third cycle implementation was not "teaching to the test", but focused on the same conversion problems as in the two previous implementations. The results of all tests are shown in Figure 2, where the lowest level corresponds to 0 points, the next level 1 point, and so on, up to the maximum level 17 points. A dashed line indicates that the student did not take one of the two tests.

It should be noted that the results improve only slightly from the pre-test to post-test 1. The median for difference in test scores is 2 points. Between post-test 1 and post-test 2, the median is 0 points. The major improvement comes, as already mentioned, for the 13 students between post-test 2 and post-test 3 where the median is 8 points.

Based on the criterion that 12 points in any of the first two post-tests is considered satisfactory, we can distinguish three groups of students: the 14 students who had satisfactory results already on the pre-test, an additional 6 students who achieved satisfactory results on either post-test 1 or post-test 2, and the remaining 13 students who did not reach the 12 points on either test. (Due to the limited effect of the second implementation, we do not distinguish between students who achieved satisfactorily on the first and second post-test.) We report the test scores for these three groups separately, in line diagrams (Fig. 3).

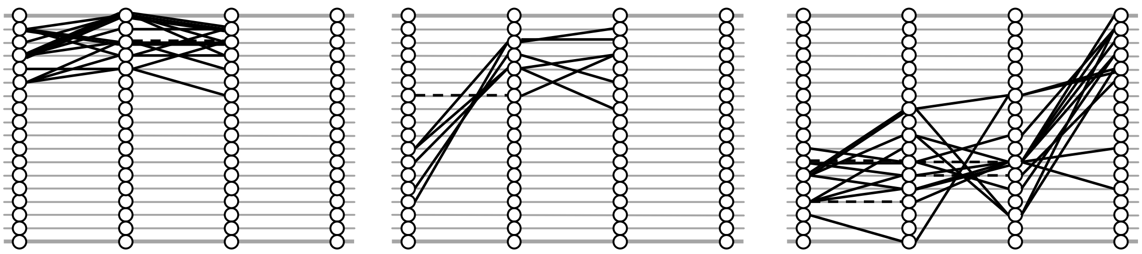

Figure 3. Test results for the three subgroups of students.

We can readily see (Fig. 3, left pane) that the 14 students who scored well already on the pre-test improved slightly on the first post-test, but their scores dropped slightly on the second post-test. Similarly, the 6 students who achieved 12 points or more on the first post-test (Fig. 3, middle pane) show a similar lack of improvement on post-test 2. The remaining 13 students (Fig. 3, right pane) show slightly improved scores on the first and second post-tests, but the substantial improvement came on the third post-test.

In retrospect, it seems as if the second teaching session did not add much for any group, while the first session resulted in a substantial improvement for 5 students (the sixth student was absent the first session but achieved 14 points on post-test 2). The third session contributed to making 11 out of 13 students achieve satisfactorily, with a median improvement of 8 points. Overall, 31 out of

33 students achieved 12 points or more on at least one of the post-tests. The major improvements occurred on the first and third post-tests (Fig. 4).

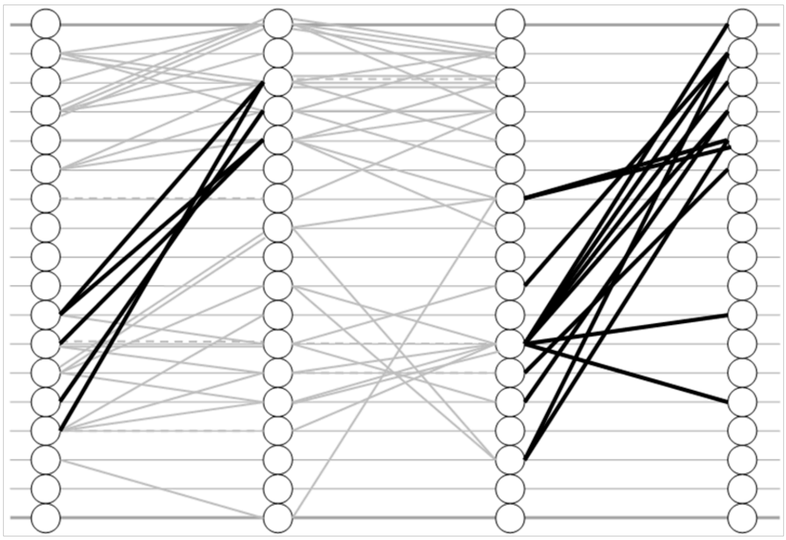

Figure 4. Highlighting improved results for two groups of students.

Discussion

Our principle-based approach, particularly the flexible adaptation of theoretical principles to emerging needs in the design process, has contributed to providing new and unexpected insights into a well-known problem, namely how we can teach students to select and use efficient strategies for subtraction. If we instead would have committed to work with a pre-defined theoretical framework, we would probably not have been able to obtain the reported findings. We have illustrated that involving principles of self-regulation, implemented through teacher-led structured activities in small groups, lead to substantial improvements of test scores for low achieving students. The introduction of this particular theory depended both on the particular context, suggestions from the teachers, and the researcher's "improvised" judgment of an appropriate treatment for the low achievers. The theory and its possible implementation were negotiated with the teachers, before they engaged in the detailed planning process. Although theories of self-regulated learning have previously been recognized as being relevant for mathematical problem solving, we could not foresee the substantial positive effects of a treatment based on scaffolding self-regulation.

However, we readily acknowledge that involving self-regulation principles was only one part of the treatment in the third cycle. In order to address the low-achieving students' cognitive and contextual processes, particularly during the phase fore-thought, planning and activation, we decided to change the pedagogical arrangements and implement teacher-led structured activities in small groups. Although it may be argued that any teacher can do a better job under such favourable conditions, the teacher still has to arrange "good" activities for the students. In the current study, our strategy has been to characterize such a good activity, for a particular group of students and their teachers, in terms of principles that guide the teachers' planning and implementation of the activity.

While classroom-based design research is often interpreted as describing recommended practices, our principle-based approach avoids this replica trap by completely avoiding descriptions of the classroom activities. Instead, we invite teachers to plan and implement teaching activities based on confirmed theoretical principles. This may be a fundamental issue for research dissemination in the learning sciences. Encouraging teachers to identify and carbon-copy so called "best practices" draws focus away from designing even better practices and may impede further improvement. Furthermore, copying practices without being informed about underlying principles may cause instability and possibly complete loss of focus on part of the teacher if the implemented activities do not proceed as intended. Rather than attempting to encapsulate current teaching practices as static recommendations for the future, we suggest a dynamic process of professional improvement based on flexible adaptation of confirmed theoretical principles. The limited involvement of researchers in a principle-based design process allows schools to involve researchers at a reasonable cost and could also stimulate a substantial number of similar studies. With maturation, such an approach could result in the encapsulation not of best practices but of best principles, not as a general set of principles for all learning objects (c.f. Kirschner, Sweller, and Clark, 2006) but a few core principles for each (type of) learning object (Dede, 2006).

Identifying theoretical principles that meet the demands in a complex design process is not an easy task. The selection of theoretical principles necessarily depends on the researcher's theoretical preferences, understanding of relevant principles, and available resources such as literature and colleagues. Despite the inherent subjectivity in the principle-based approach, we believe it is important that researchers sometimes go beyond neatly organized research programmes and engage in rather unstructured exploration of the authentic problems that teachers face in their classrooms. In our case, we are satisfied in having designed a treatment for learning subtraction strategies that proved to be successful for 33 students and their three teachers at a school in Sweden.

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Number By Reasoning and Representations – The Design And Theory Of An Intervention Program For Preschool Class In Sweden

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We describe the design process for an intervention program in the domain of number in Swedish preschool class. A consequence of the design-feedback cycle was that the initial idea of combining a learning trajectory based approach with a socially driven teaching based on collective reasoning was revised. The resulting design keeps the emphasis on structured sequences of activities and children's and teacher's reasoning about representations, but moves the learning goals from individual sessions within the program to the level of the intervention as a whole.

Introduction

Preschool class has a unique position in the Swedish education system as the bridge between the informal learning that dominates in preschool, and the formal learning following in school. It is non obligatory but in practice almost all six year old children participate. This makes preschool class a potential arena for giving children opportunities to develop skills in mathematics to remedy mathematical difficulties and remove barriers for learning. In our discussions with preschool-class teachers, they often emphasize their need for support to develop mathematics instruction and to take advantage of findings from research. The purpose of the study, partially reported here, is to design and evaluate a mathematics intervention program in the Swedish preschool class built on structured instruction design, a concrete-representational-abstract learning pattern and children's collective reasoning. The overall effect of the intervention is measured on the level of children's learning by means of a cluster randomized control study reported in a forthcoming article (Sterner, Wolff & Helenius, manuscript). The present article deals with the design phase of this study, where the purpose is to fine tune the three guiding principles into a working practical realization. The research question is: Is it possible to combine such principles into a functional program, and if so, how can such a program be described? Hence this paper methodologically falls under the design research paradigm (Edelson, 2002). McKenney and Reeves (2012) argue that educational design research is based on five intertwined principles. They are:

> *Theoretically oriented.* Empirical testing is used to validate, refine, or refute hypotheses and conjectures that are embodied in the design.
>
> *Interventionist:* Educational design research strives to produce new theoretical understanding, to positively impact practice, bringing about transformation through the design and use of solutions to real problems.
>
> *Collaborative:* Educational design research is conducted in collaboration among a range of actors and educational contexts.
>
> *Responsively grounded:* The products of educational design research are shaped by participant expertise, literature, and especially field testing.
>
> *Iterative:* The insights and the interventions of educational design research evolve over time through multiple iterations of investigation, development testing, and refinement (pp 13-15)

The work reported here honor these five principles. It is *iterative* since the design and its implementation has been tested and developed over four feedback cycles. It is *collaborative* since researchers and practitioners with different background contributed both to design, evaluation, development and theorizing of the result. The obtained theoretical principles are implemented in a teacher's handbook, available for preschool teachers and this makes the work distinctively *interventionist*. What will be mainly emphasized in this article is the *theoretical orientation* namely, three initial design principles that built on different areas of research and theory, and was embodied in a specific teaching sequence presented in the teachers handbook. We will describe how both the embodiment – the actual teacher instructions – as well as the grounding principles changed as a result of how it was *responsively grounded,* through several cycles of field testing and additional consulting with the literature.

Background

Preschool children's mathematical knowledge when starting school is highly predictive of their later success in mathematics in compulsory school (Duncan et al., 2007). Children who start school with weak mathematical knowledge tend to experience further difficulties in a downward spiral (Morgan, Farkas & Wu, 2009; Geary, 2011). In recent years there has been a growing interest in early intervention in mathematics. A meta-analysis (Diamond, Justice, Siegler & Snyder, 2013) shows that interventions vary a lot regarding the mathematical content. Examples of targeted content include: *Relational arithmetic skills* e.g. seriation, classification and conservation of numbers (Malabonga et al., 1995), *counting and efficient counting strategies, addition and subtraction with objects/pictures, add one, subtract one, estimate numbers, read and write numbers* (Clark et al., 2011), and *number line estimation* (Ramani & Siegler, 2008). There are a few studies explicitly focusing on *number sense related to reasoning about numbers* (e.g. Nunes et al., 2007; Aunio, Hautamäki & Van Luit, 2005). Math-oriented early childhood curricula have been developed in collaboration between researchers and teachers, e.g. *Number Worlds* (Griffin, 2003; 2007) focusing on the central conceptual structure of whole numbers developed by Case and Okamoto (1996) and *Building Blocks* (Clements & Sarama, 2007; Clements et al, 2011). The program *Building Blocks* focuses both on numbers and geometry and a particular feature of this program is that each domain is structured along a research-based hypothesized hierarchical learning trajectory. The theory of hypothetical learning trajectories (HLTs) is usually connected to developmental and cognitive psychology and, more recently, developmental neuroscience (Consortium for Policy Research in Education, 2011; Simon, 1995). Typically, learning trajectories connects a theoretical idea about a particular learning process leading to some learning goal, as well as practical activities designed to take the learner through the process. One of the early proponents of learning trajectories define them as “made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning processes – a prediction of how the students’ thinking and understanding will evolve in the context of the learning activities” (Simon, 1995, p. 136). Instruction and instructional programs based on learning trajectories have often proved successful and in particular several intervention programs for preschool builds on learning trajectories (see Clements et al., 2011, for an overview).

Design process

The participants in the initial design process were researchers within the psychological and mathematical disciplines and other experts on mathematics education. Here we will describe what design principles the program built on and how these principles were realized in concrete instructions to the teachers. We then describe four testing cycles and how the feedback influenced design choices and the realization of them.

Initial design principles

The first design principle is that children should be provided with a structured sequence of activities. This has been shown to be particularly effective for children at risk for mathematical difficulties (Gersten et al, 2009). This way of choosing and sequencing activities is similar to Learning trajectory based designs (Clements et al., 2011). To help structuring the program for teachers, the activities are grouped in five themes designed to be carried out by teachers over ten weeks: *Sorting, classifying and patterns; Numbers, counting and patterns; Part-part-whole; Number line*, *Grouping and place value.* The ordering of the themes and how the content in the cross reference between themes is mainly based on Griffins research on the conceptual structure of whole number (Griffin, 2003, 2007). Due to space limitation we do not deal further with the details of the sequencing or present individual activities here.

The second principle concerned the Concrete – Representational – Abstract (CRA) model, a linear model where teacher and pupils start working with concrete objects and gradually advances to the use of visual representations and further on to abstract symbols (Witzel, Mercer & Miller, 2003). In terms of learning trajectory theory, each session contained elements designed to take children from a concrete manipulation stage through several phases of representations with for example dots, squares and other icons and towards some form of symbolic or abstract reasoning with symbols like written numerals. The effectiveness of teaching mathematics through a CRA sequence of instruction to students is well documented in the literature (e.g. Allsopp, 2007; Baroody, 1987; Clarke et al., 2011; Wintzell, 2003).

The third principle involved using children's reasoning about their work and about their documentation (drawings) of their work as the main vehicle for learning. In Vygotsky's theory (1978) the social interaction between children and adult is the main source for the development of advanced mental functions. All development in the child appears first at a social and then at an individual level. Language is viewed both as a cultural tool to develop and share knowledge within a social community, and as a psychological tool to structure the processes and content of one's own thinking. Examples of cultural tools are language, art, writing, numbering etc. (Vygotsky, 1978). Drawing on Vygotsky´s work Brooks (2005; 2009) argue that when drawing is used in a collaborative and communicative manner it exists at an interpersonal level. In our design, whole class collaboration and partner work function as activities on the social level while children's drawing also at one point function on an individual level. An underlying assumption here is that drawing facilitates children's reflection on the mathematical content they previously worked on in collaboration with teacher and peers, but from a different perspective, and that the interaction between the collective and the individual, contributes to the development of thinking (Vygotsky, 1978). Children's drawings are creative representations that connect back to the collective reality they were previously engaged in. In the follow-up activity their drawings once again turn into an activity on the social level. In the discussions about their drawings each child brings a personal dimension to the enterprise. No children are alike and even if the messages being transmitted can be considered the same, it will be perceived slightly different because the receivers are different (Bishop, 1991).

Realization of design principles

To make the design principles into a teachable program, we developed a "teacher's guide" (Sterner, Wallby & Helenius, 2014). The first principle was realized by means of organizing the guide in the themes and for each theme give concrete and explicit instruction of activities to carry out. Each theme involves around ten sets of activities (sessions). The mathematics sessions were organized in a structure with six phases:

- *Counting rhymes:* A lesson starts with children and teacher gathering in a circle on the floor, counting in chorus up and down on the counting string. When a child, standing in the middle of the circle, pointing rhythmically at each child while all count together, the circle that children and teacher form is the very representation of the counting (Freudenthal, 1991).
- *Initial activity:* The teacher introduces the current task and the work is done collectively in class by using concrete objects like blocks, sticks, buttons, dices,
- *Partner work:* Children then work with partners or in small groups on similar and extended activities as they did earlier in class, using different objects or other representations.
- *Whole-class discussion:* Children and teachers come together to a joint monitoring and discussion of pair work.
- *Children's documentation:* Children create drawings as documentations of what they have done so far. The drawings are new representations that form the basis for future collective activities and discussions with teachers and peers in the next phase.
- *Follow-up activity:* Children's drawings are the starting point for further reasoning about the concepts they have worked on and connections, differences and similarities among the representations of those concepts.

Through these phases the CRA principle is realized by means of the initial work with concrete objects followed by subsequent representations of those objects when the children make their documentations. In the discussion phase, even if a child does not have an abstract idea about some concept targeted in the session, the teacher can use other children's reasoning and representations to shift the discussion towards the abstract. This means that a drawing that, from the child's point of view, started out as a representation of concrete objects and relations, may be discussed by others as a representation of abstract structures or concepts helping all children to extend their understanding towards the conceptual. In this sense, the way that the CRA model is realized in the six phases is intended to interplay with the third principle concerning the role of reasoning and social interaction.

First testing cycle.

In the first phase of the iterative stage of the design process, sessions and themes were tested by six preschool class teachers and children in their classes. One thing we learned from the collaboration with the teachers in the first cycle was to carefully choose the concrete materials to be used in the activities. In one activity the children are expected to investigate and reason about how to move soft toys between delimited quantities in order to make those quantities equivalent. The teacher experienced that the activity did not work at all since children's attention was drawn to the soft toys – everyone wanted as many as possible and they forgot all about solving the number problem. We later found this phenomenon described in the research literature (DeLoache, 2000). The more children are attracted to the physical attributes of the representation the harder it seems to be to see the symbolic information and to stick with that. In terms of our principles, this relates to the realization of CRA-principle in relation to principle of explicit structured activities. For the "C-phase" in CRA to increase the possibility of discernment of the abstract structures that are built into the particular activity, the objects should not have attractive physical or emotional attributes.

Second and third testing cycle

In the second, and later also in the third, cycle six new teachers were recruited to the team. In both cycles, a researcher (the first author of this paper) and the teachers met at seven seminars where the mathematical content and the teaching strategies were discussed. In the time between those seminars the teacher tried out the activities in their classes and documented their experiences. Teacher's documentation then became the basis for in-depth discussions at the following seminar.

A problem that emerged during the second cycle was difficulties to make all children to participate in the discussions, to express their views and suggest solutions. The teachers felt uncertain on how to pose open questions that would take the discussions and children's thinking further. We decided to complement the material with examples of questions such as: How do we know that...? What is similar and what is different in these solutions? How do we know that we have found all solutions? What will happen if we change...? How do you think Thomas thought when he made this pattern? More importantly, we also introduced a puppet into the pedagogy that sometimes came and asked questions and contributed to the reasoning in the group. The puppet has at least three equal important functions:

1. Children's ability to imagine the puppet as a "real" person help to bring out the playfulness in mathematics and "trick" them to teach the puppet and express their own views.
2. The puppet asks questions and makes statements that triggers the children's desire to reason about concepts and relationships between concepts, come up with hypothesis, provide explanations and propose solutions.
3. Using the puppet's questions and statements, the teacher can help children turn their attention to certain mathematical aspects and phenomena.

Using a puppet in the pedagogy in this way has previously been described in research (Freeman, Antonuccia and Lewis, 2000). This change related to how our third principle about children's reasoning was realized in the teacher instructions in the handbook. We concluded that for the reasoning sessions to be productive, the handbook did not only need to contain explicit activities, but also explicit tools and routines that could support teachers to carry out productive reasoning sessions.

Stage 4 analysis

In the fourth cycle eight teachers participated. Seminars were conducted in a similar manner as in stage 2 and 3. It was not until now it became apparent to us that teachers felt frustrated and uncertain of how to proceed with a subsequent session when all children did not reach what the teachers perceived as the learning goals of the present session. For example, when children documented their experiences from the work on part-part-whole relations of number seven, some children visualized the combinations by making drawings of concrete objects in two colors in different combinations. Other children drew the combinations by using dot number patterns and still others used mathematical symbols to represent different combinations like “7 0” and “6 1” with an empty space between the numerals for each combination.

On the one hand the problem seemed to be that some children when expected to use e.g. dots, circles in the representational phase, they preferred to use abstract symbols like numerals that belonged to the abstract phase. On the other hand the teacher had an idea of the group moving through the representations all together in an attempt to make sure that each child in the end reached the abstract phase and abstract understanding of every concept they had worked on. The difficulties that the teachers experienced was: 1. Children did not reach the abstract level at the same time or some children kept on using iconic representations for a long time. 2. Some children spontaneously used abstract mathematical symbols during the representational phase and the teachers meant it simply wasn’t tenable to tell the children that they had to wait to the abstract phase before they could use mathematical symbols and to share their ideas with peers.

In our discussion with the teachers we decided that instead of making sure that all the children reached a particular goal at the end of a session, it was emphasized that the primary role of the teachers was to make sure each child got opportunity to present their own representations of the activity, and have it and it’s relation to other children’s representations reasoned about in the group. In this way children's differing views and ways of expressing themselves about the activity and the concepts that were in focus in a particular session became an asset in the discussion. It was also emphasized that the relations between the mathematical themes, meant that the concepts children met were reinvented several times in different mathematical contexts.

This adjustment effectively ties all our three principles together. In essence, we place the principle that the children should be given opportunity to reason about their representations of the activity or the concept above the principle that each session should take children through the CRA stages. But the two other principles will in fact mean we can recover also the CRA-principle. Due to the emphasis on collective reasoning, even children that did not themselves reach the abstract stage in a particular session, will be part of a discussion where abstract ideas are represented. Moreover, the sequencing of the session means that the same concept is handled many times, so children will get further possibilities to reach the abstract level through the program.

Discussion

This study used both literature and experience to investigate how three design principles could be combined to support teaching mathematics in preschool class in Sweden. Findings from the field testing confirmed that these principles offered relevant support but also revealed some challenges. The first testing cycle made us make the description of activities more detailed with respect to exactly what objects to use to increase the possibilities for children to attend to the underlying abstract structures of the activity. The second/third testing cycle made us complete the teacher material with more detailed instructions, tools and routines for how to make the reasoning sessions more productive. Both these changes concerned the embodiment of the CRA principle and the principle of reasoning about representations respectively.

The discovery in the fourth testing cycle however, was of a different nature. As pointed out, our program has many similarities with a learning trajectory design. Even though it is not required theoretically, in such designs individual activities often come with learning goals. In addition to the structured design, our program build on sessions involving collective reasoning about children's individual representations of collectively experienced activities. In our testing, we found that the idea of sequenced learning goals tied to such sessions created a conflict with the idea of collectiveness. When rethinking our design, we concluded that each session was better seen as an instance to get a particular type of experience.

This design is the result of work both from researchers as well as from teachers and their children. The effectiveness in terms of overall student outcomes is currently analysed. It would be an interesting exercise for future design work and research to examine in what sense these design principles are transferable to other contexts, like other areas of mathematics or other ages of students.

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The Impact of a Professional Development Program in Formative Assessment on Teachers' Practice and Students' Achievement

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There is research evidence for the impact of formative assessment, but not for how to support its implementation. From viewing formative assessment as one big idea and five key strategies, the effects of a professional development program (PDP) in formative assessment on teachers' classroom practice and students' performance in mathematics were investigated by using classroom observations, interviews, questionnaires, and student mathematics pre and posttest. Based on results of major changes in classroom practice and statistically significant ($p<0.05$) effect on student mathematics performance ($d=0.66$) these studies can contribute to knowledge about how to design PDPs in formative assessment.

The Importance Of Grammatical Style In Mathematics Tests For Second Language Learners And Low Performing Students

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The study is a work in progress which aims at investigating how different grammatical styles in the verbal language in mathematics test tasks in TIMSS 2011 might be of importance for the performance of low achieving students and second language learners. The method that will be used in the study is a correlation analysis between grammatical style and students´ test scores. Some tentative results show that different grammatical styles are used differently in the various content areas of the subject. The results also show that an academic style is of importance for the achievement of both groups of student, although there are differences between different content areas.

A modelling approach for teaching statistics and probability

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Recent research on students' reasoning during the process of statistical inference (SI) has highlighted an informal way to approach SI in statistics teaching. This informal approach, informal statistical inference (ISI), might be seen as either a skill for the statistically literate citizen or as the root of understanding of formal inference. Research has found generalization, using data as evidence and probabilistic language as three key aspects that describes a successful process of ISI. The basic idea of this presented research is that statistics teaching should be viewed as data modelling - developed in the context of real world situations where students make inference from unknown distributions. A developed framework, known as ISI-modelling, is tested in this study.

The Impact of a Professional Development Program in Formative Assessment on Mathematics Teachers' Classroom Practice

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This study is a sub study in a project about a comprehensive professional development program (PDP) for mathematics teachers in formative assessment (FA). My aim is to investigate in which ways the participating teachers' classroom practice change, due to the delivered PDP, and also to identify reasons for the changes and the variation in changes. Fourteen randomly chosen mathematics teachers in secondary school participated in the PDP. The teachers were interviewed and their classroom practice observed before and after the PDP. They have also answered two questionnaires. Preliminary results show that all teachers were motivated to change and did change their practice, but to varying degrees.

The Quality of Supervised Group Discussions within the Frame of Cooperative Learning

Gerd Brandell
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This presentation reports preliminary results from an on-going research project, based on classroom observations. The study is done in collaboration with the teachers involved. The aim of the study is to describe the quality of the discussions among engineering students in their second year learning calculus of several variables through co-operative work in small groups. The empirical material consists of a series of video-recorded lessons analysed within a theoretical framework from the anthropological theory of didactics. The results show that the students spend time both on techniques for solving various types of tasks and on issues related to the "theory block".

Multiple Mathematical Practises Figuring in a Lecture About Assessment

Andreas Ebbelind
Linnaeus University, Sweden

This presentation problematizes how a lecturer's prior experiences contribute to immediate interaction during a lecture about assessment. The conceptual framework Patterns of Participation guided the study and data was structured through Systemic Functional Linguistics. During the lecture the lecturer intertwined several personal narratives related to personal experience for example: using textbook, dissatisfaction with former steering documentation, pupils' different learning strategies and how to teach professionally. However, these narratives raise questions, for example: how personal narratives and official discourses of assessment are established, the relationship and function they have in relation to student teachers understanding of assessment.

Student teachers' reasoning about the mathematical content in pupils' solutions

Birgit Gustafsson
Mid Sweden University

In this ongoing study the focus is to investigate student teachers' interpretation of the mathematical content in first year upper secondary school pupils' solution of two algebraic problem solving task. What characterizes the student teachers' communication is of special interest. To investigate these two analyse methods are available, the epistemological triangle (Steinbring, 2006) and the IC-Model (Alrø & Skovsmose, 2002). The epistemological triangle will give focus upon the relations between object, sign and concept. With the IC-Model the focus will be on the interaction between the student teachers. But is it possible to combine the to analyse methods? What are the obstacles?

Compulsory School Students' Experiences of Mathematic Teachers' Assessment Practice with a Focus on Communication

Lena Heikka
Lulea University of Technology, Department of Arts, Communication and Education

The aim is to explore Swedish upper elementary school students' experiences of mathematics teachers' assessment practice, using ethnographic methods to investigate multiple cases, with a focus on teacher's communication about learning goals. The overarching research questions are; how do mathematics teachers communicate their assessment practices and how are they perceived by the students? Preliminary results show that teachers and students express a lack of knowledge about the syllabus in mathematics. Observations in the classroom confirm that the students are poorly informed, especially regarding the knowledge requirements. Student's makes strong connections between the content of the chapter in the textbook and the goals of the teaching of mathematics.

Mathematics Communication within the Frame of Supplemental Instruction SOLO & ATD Progression

Annalena Holm
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Supplemental Instruction (SI) is considered a method for cooperative/collaborative learning being used at universities in many countries including Sweden, and lately also in upper secondary schools. Several studies have been made to evaluate SI in universities throughout the world, but hardly any studies have been made at lower levels. This project aimed at developing an analysis strategy that was a combination of ATD and the SOLO-taxonomy. The aim was also to use the strategy when analysing students' discussions in mathematics at SI-sessions in Swedish upper secondary school.

Young Pupils' Way of Explaining and Arguing in the Discourse of Mathematics

Eva Juhlin
Luleå University of Technology, Sweden

Language play an important role in the development of mathematical concepts and mathematics could be seen as a communicative discourse. This implies that communication is an important tool in becoming a competent actor in the discourse of mathematics. The text presented is a short report of an ongoing study with the aim of studying young students' communication in the discourse of mathematics in a Swedish mathematical classroom focusing on geometry.

From Natural Numbers to Integers ($N \rightarrow Z$) – A Learning Study about the Importance of Identifying Critical Aspects to Enhance Pupils´ Learning

Anna Lövström
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The thesis is focused on producing knowledge about which distinctions are necessary for pupils to do, in order to extend their number domain from natural numbers (N) to integers (Z). The thesis adopts the theoretical framework Theory of Variation (Pang & Marton, 2003), which makes it possible to analyse the relation between teaching and learning in commensurable terms. Data is collected from two Learning Studies. Participating pupils are eight to nine years old. Preliminary results indicate, for example, that pupils tend to use subtraction according to the commutative law of addition. They don´t seem to discern the difference between 3-2 and 2-3. That makes it difficult for pupils to realize the need for negative numbers.

Tools for teachers – The issue of developing mapping tests for primary school mathematics

Guri A. Nortvedt and Andreas Pettersen
University of Oslo

In 2008, Norway implemented a mandatory national mapping test to help Grade 2 teachers screen their students in mathematics. Informed by recent research on mathematics learning difficulties and experiences from the first generation tests, a second generation of tests was developed. Two tests for each grade level (1 – 3), were piloted with national representative samples (N=550 – 600). Analysis reveal that the item format plays a crucial role; number size, number line labelling, and object grouping influence the difficulty level of the item considerably. When tests are to be targeted to the lowest performing quintile group, this result must be considered in item and test development as items needs to be tailored to the students in question.

Negotiating mathematics teaching? A study of a mathematics teacher's agency in collegial collaboration

Anna Pansell
Stockholm University

The context of this short presentation is a case study of one teacher, Mary, focused on understanding complexity of mathematics teaching. The analysis shows how Mary negotiates her role as a teacher of mathematics meeting her colleagues. In a decentralised school system teachers are at the centre stage of curriculum enactment (Skott, 2004), renegotiating the relationship between the pedagogic and the official recontextualising field (Bernstein, 2000). The study addresses how notions of agency can facilitate analyses of teacher's collegial negotiations of mathematics teaching. Reading transcripts of recorded teacher meetings following how Mary exhibits agency it was possible to capture recontextualisations. A combination of agency and recontextualisation could help capture the essence of why mathematics teachers teach the way they do.

Developing Mathematics Instruction with Adaptive Conceptual Frameworks

Miguel Perez and Håkan Sollervall
Linnaeus University

In this paper, we elaborate on a flexible strategy based on the development of adaptive conceptual frameworks that we have used to conduct design-based research with the aim of developing ICT supported mathematics instruction. We differentiate between Conceptual Frameworks for Development (CFD) and Conceptual Frameworks for Understanding (CFU) depending on how the frameworks are used in the design process. In this approach, we connect empirical data with confirmed theories in an adaptive and iterative process. We believe that adopting such a flexible approach allows us to fully make use of our available resources to address authentic educational needs as expressed by practicing teachers.

Algebra Tasks in a Word Problem and Non-Word Problem Context – A Multilingual Project

Jöran Petersson, Eva Norén
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Many students see algebra as a difficult topic. For second language speakers there might also be difficulties comprehending an algebra task linguistically correct. The authors suggest studying how knowledge in algebra and linguistic registers in mathematics interplay for both newly early arrived immigrants compared to first language speakers. We suggest the tools for such a study to be measuring achievement and solution strategy while varying the text intensity and mathematics register in algebra problems for students with different length of experience of Swedish language in school year 9. We want to discuss design of test instrument and methods for background data collection.

Conventions as Obstacles for Understanding? – Pupils' Reasoning when Making Sense of School Mathematics Language

Elisabeth Rystedt
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This study focuses on a group of pupils when reasoning about a task including the algebraic convention to answer "expressed in n". (In Swedish: "uttryckt i n".) The aim is to investigate what resources the pupils make use of and how they apply and understand the colloquial, inter- and school mathematics language. The study builds on video graphed classroom data. The analysis is currently underway, but preliminary conclusions point to that the generalisation itself is not problematic. The obstacle for understanding is how to interpret the convention to answer "expressed in n" (school mathematics language).

Integrating writing to support students´ understanding of reading in mathematics

Cecilia Segerby
Malmö University

Resent research shows that the dominant practice in mathematics education in Sweden involves students reading and working individually in a textbook. However, to read mathematical texts means understanding the global meaning from the page which requires specific reading skills. In this study, which is a part of a larger educational design study, specific writing activities are implemented into a Year 4 class to support the students´ understanding of what they read in their textbook in mathematics. The study is based on Palinscar and Brown´s reciprocal teaching activities clarification and summarization which are connected to Halliday´s Systemic Functional Linguistics.

Kompetensutveckling i matematik för pedagoger i förskola och förskoleklass

Christina Svensson, Troels Lange, Anna Wernberg
Malmö högskola

Studien är ett bidrag till forskningen inom matematikdidaktiken om kompetensutveckling i matematikämnet för pedagoger inom förskola och förskoleklass. I den här korta presentationen föreslås tre frågeställningar för diskussion utifrån genomförande av en fallstudie för att synliggöra hur pedagogers matematikundervisning i förskola och förskoleklass förändras över tid under en kompetensutveckling baserad på Bishops (1988) sex matematiska aktiviteter.

Immigrant Students' Perspective On Learning Mathematics

Petra Svensson
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The study presented is examining how immigrant students, who live and attend school in multicultural and socially deprived areas, are experiencing their possibilities to learn mathematics. The thesis discusses how public discourses affect students' foregrounds and rationales for learning in different ways, contributing to how the students perceive their possibilities to learn mathematics. The empirical material consists of focus group interviews with students in grade 9. The results show that immigrant students' shortcomings in mathematics are affected by wide variety of influences, most of which are out of their control to change. The study suggests that it is of importance to consider the influence of public discourse on students' possibilities to learn mathematics.

Communicating Mathematically with Images

Anna Teledahl and Eva Taflin
Dalarna University

This short presentation is part of a PhD project that focuses on students' writing in mathematics. The study presented here takes a starting point in the idea that students use different semiotic resources when they represent their different ideas in writing and that these resources offer different opportunities for communicating. A sample of 300+ accounts of mathematical problem solving was collected from 6 different groups of students aged 10-11. The analysis was focused on identifying the way in which images were used to communicate. Preliminary findings indicate that iconic, symbolic and illustrative images were used. A majority of these were manipulated in different ways, which suggests that they assisted the student in the problem-solving process, that the student used the manipulation to show their process or both.

The Discursive Use Of Gestures In University Mathematics Lecturing

Olov Viirman
University of Gävle & Karlstad University

In a number of publications I have used the commognitive framework to investigate various aspects of the discursive practices of seven university mathematics teachers from three different Swedish universities. I now intend to use the same set of data to study the teachers' use of visual mediators, and in particular gestures, in their lecturing, using a semiotic approach as a complementary theoretical perspective. No systematic analysis has been conducted as yet, but observations made during the analyses conducted for the previous papers suggest that there are specific gestures associated with certain types of discursive actions, and with certain types of diagrams. For instance, domain-range diagrams of functions are often accompanied by a sweeping motion of one hand along the arrow, suggesting motion from domain to range; this in turn could be seen as indicative of a process view of functions.

How Hard Can It Be? What Knowledge and Skills Does a Teacher Practising Formative Assessment Use?

Charlotta Vingsle
Umeå Mathematics Education Research Centre, Umeå University

This case study investigates a teacher's use of knowledge and skills during interaction in whole-class while using formative assessment. This practice includes eliciting information about student learning, interpreting the responses, and modifying teaching and learning activities based on the given information. The teacher has participated in a professional development program in formative assessment and teaches grade 5 in mathematics. The results of the study will be presented at the conference.

The meaning of concept

Lotta Wedman
University of Gothenburg: Department of Pedagogical, Curricular and Professional Studies

There is a need, both for teachers and for researchers in mathematics education, for a discussion about the meaning of the word concept. The conceptual analysis in my work uses both philosophy and mathematics education to build a base for this discussion. I raise questions like if it's possible for us to agree about a common view of concepts or if it's feasible for us to have different views.

Valuing Mathematics: Translation Challenges

Lisa Österling and Annica Andersson
Stockholm University

In this presentation, we share our experiences of the linguistic and cultural adaptation of a questionnaire, exploring what students value as important when learning mathematics. From the pilot test of the translated questionnaire, we found that translation and back-translation was not enough to ensure metric equivalence when translating a questionnaire from English to Swedish language. We faced a cultural difference in the meaningfulness of items in the questionnaire. This difference consisted of both what mathematical content students could recognize, as well as what activities from the mathematics classroom students were familiar with. As a result, the questionnaire was submitted to a cultural adaptation and thereby more meaningful for respondents.

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