The use of DGE and CAS to support mathematical thinking competency: a literature review

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Focusing on the potential that dynamic geometry environments (DGE) and computer algebra systems (CAS) offer for mathematical inquiries, this paper presents a literature review of the use of DGE and CAS in relation to the mathematical thinking competency (MTC) of the Danish competency framework (KOM). This specific competency concerns modes of thinking when engaging in mathematical inquiry. The 17 studies included in the review were analysed from the perspective of MTC, resulting in the identification of three ways to use DGE and CAS as tools in activities related to the MTC.

Over the last three decades, the extensive incorporation of digital technologies in schools has become an important and well-researched part of mathematics education research (Artigue, 2010; Trouche et al., 2012; Villa-Ochoa & Suárez-Téllez, 2021). During the 1980s, digital technologies were predicted great potential for enhancing the teaching and learning of mathematics. However, years of research on the use of digital technologies in mathematics education have identified limitations and new difficulties (Jankvist & Misfeldt, 2015; Niss, 2016). One reason may be that tasks originally developed for paper and pencil have simply been adapted to digital environments without taking into account the different epistemic values of working with paper and pencil and within digital environments (Artigue, 2010). Nowadays, specific features of different digital tools for mathematics are integrated into one another. For instance, tools mainly functioning as dynamic geometry environments (DGE) also include spreadsheets and computer algebra systems (CAS) and vice versa (e.g. GeoGebra www.geogebra.org and TI-Nspire CX CAS student software https://education.ti.com).

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Furthermore, different notions of mathematical competence, proficiency, literacy and similar terms, have addressed the focus on mastering mathematics and mathematics practice in comparison to earlier focus on skills and knowledge (Niss et al., 2016). Yet, the interplay between the use of digital technologies and mathematical competence has only recently drawn attention (e.g. Geraniou & Jankvist, 2019).

In Denmark, the notion of mathematical competence comprises the Danish competency framework (KOM) (Niss & Højgaard, 2011) – a description of eight mathematical competencies of doing and dealing with mathematics – which has been implemented in school curricula for primary, secondary and tertiary education. It served as the foundation for the PISA 2015 assessment and analytical framework for mathematics (OECD, 2017), inspiring mathematics education in other parts of the world (Niss et al., 2016), e.g. Sweden in Scandinavia (Boesen et al., 2014). Thus, it has become relevant to study the use of digital technologies in relation to the mathematical competencies of KOM, such as the reasoning (Højsted, 2020) and the representation competencies (Pedersen et al., 2021). The mathematical thinking competency (MTC) is the focal point of the present paper and concerns someone's ability to engage in mathematical inquiry.

The descriptions of the mathematical competencies focus on the cognition of mathematical activities. Mathematics education research has addressed the use of digital technologies in relation to cognition through the lens of theoretical constructs, such as *instrumental genesis* (e.g. Guin & Trouche, 1998) and *human-with-media* (Borba & Villarreal, 2005).

This review attempts to outline the affordances of using digital tools for mathematics education, particularly in relation to the MTC. Therefore, before formulating the research question and outlining the criteria for the selection of literature, the next section explains the KOM framework and, in particular, the MTC.

The KOM framework and the MTC

The Danish KOM framework is a characterisation of what it means to master mathematics across different topics and levels. It consists of eight mathematical competencies (Niss & Højgaard, 2019), as illustrated in the so-called KOM flower in figure 1. KOM defines mathematical competency as "someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations" (Niss & Højgaard, 2019, p. 14). Four of the competencies concern asking and answering in, with and about mathematics. These are the competencies of mathematical thinking (MTC), problem handling, modelling and reasoning. The other four competencies deal with mathematical language and tools, namely the competencies of representation, symbol and formalism, communication and aids and tools. All eight are intertwined in mathematical activity, while each of them focuses on the distinct cognitive processes of mathematical activities (Niss & Højgaard, 2019).

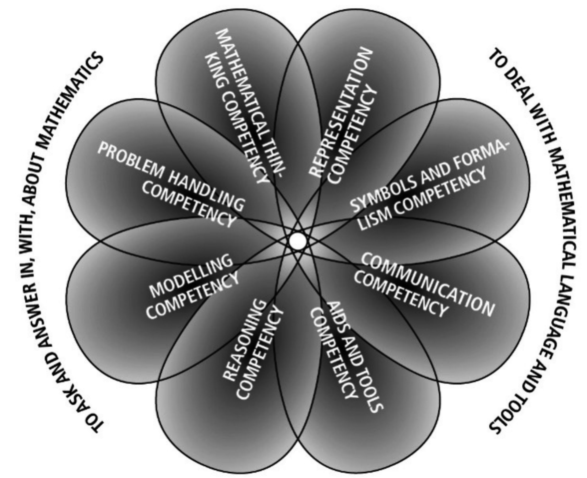

Figure 1. *The so-called KOM flower (Received from T. Højgaard, co-author of KOM)*

MTC is the ability to master the modes of thought required when engaging in mathematical inquiry (Niss & Højgaard, 2019). It concerns the nature of questions and answers mathematicians investigate and seek. It neither involves the activities of reasoning nor problem-solving and posing, but rather the reflections before and after such activities. Based on the description of the MTC, I have divided its content into four aspects (Pedersen, 2022).

The first is *the question–answer aspect*, which is the ability "to relate to and pose the *kinds* of generic questions that are characteristic of mathematics and relate to the nature of answers that may be expected to such questions" (Niss & Højgaard, 2019, p. 15).

The second, *the mathematical statements aspect*, involves "distinguishing between different types and roles of mathematical statements (including definitions, if-then claims, universal claims, existence claims, statements concerning singular cases, and conjectures), and navigating with regard to the role of logical connectives and quantifiers in such statements, be they propositions or predicates" (Niss & Højgaard, 2019, p. 15).

The third, *the scope of concept aspect*, comprises the ability to relate ”to the varying scope, within different contexts, of a mathematical concept or term” (Niss & Højgaard, 2019, p. 15).

The fourth, *the generalisation–abstraction aspect*, concerns ”relating to and proposing 'abstractions' of concepts and theories and 'generalisation' of claims (including theorems and formulae) as processes in mathematical activity” (Niss & Højgaard, 2019, p. 15).

Regarding the use of digital technologies as platforms to engage in mathematical inquiries, Niss (2016) argues that digital tools can enhance mathematical capacities as well as replace mathematical competencies. On the one hand, with easy access to different representations and shifts between them, digital technologies can replace processes of symbol manipulation and translation between representations (Pedersen et al., 2021). On the other hand, they offer platforms for investigating mathematical concepts, processes and relations in other ways than non-digital environments (Niss, 2016).

Mathematical thinking in relation to the MTC

The notion of mathematical thinking in mathematics education research is broad and often used in connection with mathematical activities, such as reasoning, problem-solving, modelling, generalisation and abstraction (e.g. Drijvers et al., 2019; Goos & Kaya, 2020; Mason et al., 2010; Tall, 1991; Villa-Ochoa & Suárez-Téllez, 2021). Recently, two literature reviews on mathematical thinking were published. Goos and Kaya (2020) focus on understanding and promoting students' mathematical thinking as the activities of problem-solving and reasoning, including processes of explanations, generalisations and abstractions, without taking the role of digital technologies into account. Villa-Ochoa and Suárez-Téllez (2021) aim to identify the contributions of DGE and CAS for mathematical thinking, focusing on the development of reasoning and mathematical modelling. In KOM, reasoning, problem handling and modelling are competencies distinct from the MTC. Thus, in contrast to the general and broad view on mathematical thinking, MTC is separate from the activities of reasoning, modelling and problem-solving, yet with overlaps.

From the perspectives of *Thinking mathematically* (Mason et al., 2010) and *Advanced mathematical thinking* (Selden & Selden, 2005; Tall, 1991), mathematical thinking consists of different thinking processes concerning different activities. Some focus on the details of questions and answers when tackling problems before the actual problem-solving process begins. Others concern the scope of the results, processes, concepts and conditions of a problem and the extension of these as part of new but similar questions and problems. This extension of questioning problems and results is connected to the processes of generalisation and abstraction (Mason et al., 2010). These kinds of processes overlap with the question–answer, scope of concept and generalisation–abstraction aspects of the MTC.

The focus in advanced mathematical thinking is on formalism going "from *describing* to *defining* and from *convincing* to *proving* in a logical manner based on definitions" (Tall, 1991, p. 20). This means that awareness of and ability to apply the logical structure of mathematics and mathematical statements are part of advanced mathematical thinking (Selden & Selden, 2005; Tall, 1991; Vinner, 1991). Moreover, Dreyfus (1991) elaborates the processes involved in advanced mathematical thinking and emphasises the processes of generalisation and abstraction. Thus, advanced mathematical thinking emphasises processes similar to the mathematical statements and generalisation–abstraction aspects of the MTC.

Research questions

With the overlaps between the general notion of mathematical thinking and the aspects of the MTC, this paper addresses the following questions:

RQ1 What aspects of MTC can be identified in the mathematics education literature explicitly addressing the interplay between the use of DGE and CAS and mathematical thinking?

RQ2 What ways of using DGE and CAS to support students' activities related to the MTC can be indicated in the included literature as part of answering RQ1?

The present paper indicates ways for DGE and CAS to function as tools for the activities related to the MTC rather than conducting an exhaustive literature review on the use of DGE and CAS to enhance mathematical thinking.

Review method

The MTC, being central to this study, naturally influenced the conceptual framework and other phases of the review, including the literature search strategy, the inclusion and exclusion criteria for the screenings, the coding of the studies and the synthesis of the results per each research question (Newman & Gough, 2020). As RQ1 presents, the set of literature for the searches is delimitated to the intersection of studies explicitly focusing on mathematical thinking and the use of DGE and CAS.

Table 1. *Overview of keywords for the searches in the databases*

	Mathematical thinking	The use of DGE and CAS
Keywords	”mathematical thinking” OR ”thinking mathematically”	ICT OR ”digital tool*” OR ”digital technolog*” OR ”dynamic software” OR ”dynamic geometry” OR DGE OR DGS OR geogebra OR CAS OR ”computer algebra” OR ”symbolic calculator” OR ”graph* calculator”

Therefore, the keywords for the search strings entered into the selected databases (cf. below) were the words related to DGE and CAS (table 1) intersected with ”mathematical thinking” or ”thinking mathematically”.

The searches took place in four databases: Web of Science, and ProQuest for searches in ERIC, the Education Database and PsycINFO (April, 2021). The searches were restricted to English language and peer-reviewed works, but otherwise, there were no restrictions regarding the type or year of publication. Covidence (covidence.org), which is an online tool for managing literature reviews, was the tool applied for the resulting list of literature. The screening followed Newman and Gough's (2020) two steps: 1) title and abstract screening and 2) full text screening. Together, the two searches yielded 130 results with 32 duplicates, leaving 98 studies for the first title and abstract screening.

MTC as inclusion and exclusion criteria

The MTC and its four aspects were scrutinised to elaborate a priori inclusion and exclusion criteria to code the studies for indications of activities related to the MTC. First, MTC focuses on different activities of mathematical inquiry and investigations, which are key elements of this competency. The four aspects of the MTC elaborate on these activities.

To code a study within the question–answer aspect of the MTC, the study should include examples or descriptions of how CAS or DGE assist mathematics learners to explore, extend and question mathematical questions and answers. This means focusing on students' reflections related to processes of problem-solving and posing but not on the actual problem-solving and posing strategies.

A study categorised as involving the mathematical statements aspect should focus on participants working with definitions or different kinds of claims. It had to include examples or descriptions of how mathematic learners consider logical connectives and/or quantifiers as part of claims, conjectures and inferences. However, justifying conjectures is outside the scope of the MTC.

For the scope of concept category, the study should include examples or descriptions of how exploring a problem or concept using CAS or DGE can encourage participants to study or extend the investigated problems or concepts in other contexts or settings. In some situations, this category may be related to the question–answer aspect of the MTC.

Finally, a study was categorised as including the generalisation–abstraction aspect if it involves examples or descriptions of mathematics learners generalising examples to more generalised concepts based on studies of invariance and contrasting, as well as abstracting to formalised concepts based on "empirical" work with examples using the tools.

Screening the literature

For the title and abstract screening, Covidence allows users to vote "Yes" (forwarded to full text screening), "No" (exclusion) and "Maybe" (forwarded to full text screening). Covidence keeps track of the historical record of each study, from being imported to the voting, meaning that the votes of "Yes" or "Maybe" can be seen during full text screening.

In the title and abstract screening, the studies were selected based on the following inclusion and exclusion criteria (table 2). Few studies

Table 2. *Overview of the inclusion and exclusion criteria for the review*

Criterion	Inclusion	Exclusion
1: Mathematical thinking	Mathematical thinking should be part of the object of study and not only a research tool used to study other aspects of mathematics education research (Goos & Kaya, 2020). In the title and abstract screening, the study should indicate that the term "mathematical thinking" applied in the study involves activities of the MTC, according to the coding categories. Keywords for inclusion: doing and dealing with mathematical inquiry and investigations; exploring, questioning and extending mathematical problems; working with definitions, claims and conjectures and their logical structures; considering the scope of mathematical concepts in different contexts and settings; and working with generalisations and abstractions.	If "mathematical thinking" is used in an unspecified manner, where the abstract of the study does not indicate activities related to the MTC aspects, cf. the mentioned keywords. Or, if the term "mathematical thinking" and the keywords are used merely in the context of problem-solving, reasoning or modelling, not including reflections related to the MTC (e.g. if a study focuses on problem-solving but not problem-posing in continuation hereof, as questioning the extension of the problem or the results).
2: DGE and CAS	DGE and/or CAS are part of the focus of the study. Here DGE and CAS include symbolic and graphic calculators and spreadsheets.	Everyday communication tools, social media and programming tools
3: Age and type of participants	Students from primary, lower secondary and upper secondary school, as well as students from first-year university and preservice and in-service teachers	Students with special needs, bilingual students, gifted students and similarly specialised foci
4: Content	Mathematical topics and content (e.g. algebra, geometry, analysis and calculus)	Programming, natural sciences, and engineering
5: Type of study	Journal papers, book chapters and conference proceedings	Book reviews, journal comments, overviews of proceedings, dissertations and inaccessible studies

did not provide an abstract and were voted "Maybe" for further full text screening to avoid excluding possibly relevant studies.

For Criterion 2, tools such as spreadsheets and symbolic and graphic calculators were considered part of the inclusion criteria as a subcategory of DGE and CAS, since these tools include features similar to features of DGE and CAS and therefore may have potential in enhancing the aspects of the MTC.

During title and abstract screening, 52 studies were found to be irrelevant based on the above criteria, leaving 46 for the full text screening (see table 3). With the database searches being the main way to access relevant literature, I conducted two additional title and abstract screenings of the two journals Mathematical Thinking and Learning (MTL) and Digital Experiences in Mathematics Education (DEME). Works in MTL were featured in searches carried out in the databases, but I found it important to conduct an additional screening because of the journal's specific aims and scope regarding mathematical thinking. DEME is a newer journal that was first launched in 2015 and was specifically chosen because only issues published beginning in 2020 are available in ERIC and could not be found in the other databases. In contrast, other prominent journals focusing on digital technologies in mathematics education (Williams & Leathem, 2017) were represented in the chosen databases. For both additional journals, the searches included the same keywords as those used in the databases. Six studies were added to the full text screening, three from each journal, resulting in 52 studies for the full text screening (see table 3).

Table 3. *Overview of the review process and the screenings*

Imported from searches in databases	130 studies 32 duplicates
Title and abstract screening	98 studies for screening 52 excluded based on criteria in table 2

Full text screening	46 studies for screening
	6 added from MTL and DEME
	52 studies for screening
	35 excluded
	    20 excluded based on Criterion 1
	    6 excluded based on Criterion 2
	    5 excluded based on Criterion 5
	    2 not found

Included	17 studies

For the full text screening, the studies were selected based on the inclusion and exclusion criteria in table 2, where the indications from the title and abstract could be confirmed (included) or rejected (excluded). Furthermore, the studies were coded using the four aspects of the MTC for the purpose of this review (i.e., to identify indications of DGE and CAS as tools for these types of mathematical activities). If none of the four aspects of the MTC could be identified, the study was excluded based on criterion 1. Of the 52 studies, 17 were included in the review. Table 3 provides an overview of the inclusion process using Covidence.

All five criteria were applied for exclusion during the process. Criteria 3 and 4 were used for exclusion during the title and abstract screening, and Criteria 1 was mostly used in the full text screening as part of the coding process of the studies. At first, studies focusing on problem-solving, reasoning or mathematical thinking were included in the title and abstract screening to be analysed in detail, given the aspects of the MTC in the full text screening. If not one of the MTC aspects could be identified as reflecting the described problem-solving, reasoning or mathematical thinking requirements, the study was then excluded during the full text screening.

Conducting the review in this way certainly has methodological limitations. The search strategy involves searching for aspects of the MTC only from the intersection of literature on ”mathematical thinking” and ”the use of DGE and CAS”. The area in which the literature was actually collected is illustrated in the Venn diagram in figure 2, illustrating the intersection of the sets of literature of interest. Literature involving aspects of the MTC not written into the thematic discussion of mathematical thinking may have been left out, which will be addressed in further detail as part of the analysis and discussion.

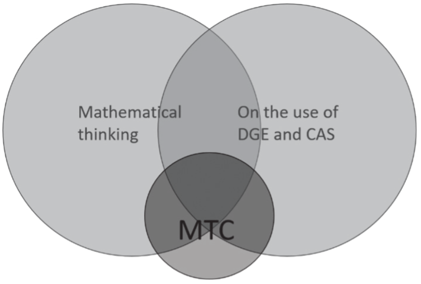

Figure 2. *The research areas involved in the review*

Analyses and results

Using the MTC as an analytical lens, the 17 studies are categorised in terms of the four aspects, and most studies address more than one of these. The question–answer, scope of concept and generalisation–abstraction aspects of the MTC are all identified in some of the studies. The mathematical statements aspect is standing out, by implicitly being involved in the studies. Moreover, the studies are classified based on tool use, where spreadsheets and graphic calculators have their own categories for later analysis in terms of DGE and CAS. One study uses two kinds of tools, namely DGE and spreadsheets (da Silva et al., 2021), why it is counted twice in the distribution between the tools. The distribution of the studies in relation to the tool in use and the aspects of the MTC is illustrated in table 4.

Table 4. *Overview of the studies categorised based on digital tool and aspects of the MTC*

Tool	MCT aspect			
	Question–answer	Mathematical statements	Scope of concept	Generalisation –abstraction
DGE (*n* = 11)	*n* = 4	-	*n* = 5	*n* = 8
CAS (*n* = 2)	*n* = 2	-	*n* = 0	*n* = 2
Spreadsheets (S) (*n* = 2)	*n* = 2	-	*n* = 1	*n* = 2
Graphic calculator (GC) (*n* = 3)	*n* = 1	-	*n* = 1	*n* = 3

The following four subsections present the analyses of the studies to identify tendencies for using DGE and CAS in relation to the respective aspects of the MTC.

The nature of mathematical questions and answers

Eight of the included studies illustrate elements connected to the question–answer aspect of the MTC ($N_{DGE} = 4$; $N_{CAS} = 2$; $N_S = 2$; $N_{GC} = 1$). The three studies (Reyes-Rodriguez et al., 2017; Santos-Trigo, 2004; Santos-Trigo & Reyes-Rodriguez, 2016) using merely DGE, all focus on the affordances of DGE in formulating questions. The dynamic features offering an explorative platform can assist in questioning the concepts, the relations and the statement of a specific problem (Santos-Trigo, 2004). Initially, problematising a task is crucial in identifying different strategies for solving a task and thereby for the processes of problem-solving (Santos-Trigo & Reyes-Rodriguez, 2016). Manouchehri (2004) argues that graphic calculators can provide representations that students can study and analyse in detail and thereby assess the given problems more carefully. This involves deeper analysis of the given questions that need to be answered to obtain the expected answer. Studying and questioning a specific task and using DGE in this way can also help mathematics learners relate to the expected answer. This is the case in a study (da Silva et al., 2021) providing an empirical example, where a group of mathematics teachers uses GeoGebra in an explorative way to understand the given question and its expected answer. Thereafter, they shift to spreadsheets in Excel, as they find this tool more appropriate for solving the given task. Another way of using spreadsheets for this aspect of the MTC is to provide an overview of a large number of calculations. From this, new questions arise, expanding the inquiries from the originally intended investigations (Calder, 2012). The studies reporting on DGE, spreadsheets and graphic calculators illustrate how these tools can play an active role in investigating further mathematical relations based on initial inquiries into a task or a problem.

The two studies focusing on CAS use (Ismail et al., 2014; Zeynivandnezhad & Bates, 2018) illustrate examples of students relating to the expected answers. When students solve a task or verify a resolution using CAS, they need to be aware of the answer they expect to interpret the outcome and make meaning of the feedback (Ismail et al., 2014; Zeynivandnezhad & Bates, 2018). This illustrates the importance of awareness of the expected answers.

These findings illustrate how digital tools can be part of mathematical exploration as student activities related to the question–answer aspect of the MCT. Moreover, the studies imply that this aspect is important as part of students' anticipation when applying digital tools. The same two-way relationship between the MTC and the use of digital tools appears of the generalisation–abstraction aspect of the MTC.

Generalisation and abstraction as processes in mathematical activity

Of the 17 studies, 14 report on the application of DGE and CAS in relation to processes of generalisation and abstraction ($N_{DGE} = 8$; $N_{CAS} = 2$; $N_S = 2$; $N_{GC} = 3$). These studies focus on generalisations of single problems or cases to establish mathematical concepts but not necessarily on the processes of the abstraction of mathematical concepts.

All studies reporting on CAS, spreadsheets and graphic calculators ($n = 7$) include elements of the generalisation–abstraction aspect. When using CAS for generalisation, and to some extent, abstraction, high-speed calculation enabled by CAS is emphasised as the main affordance. Students can generalise and conjecture after experimenting with several calculations using CAS (involving graphic tools) (Ismail et al., 2014; Zeynivandnezhad & Bates, 2018). The same goes for using graphic calculators, which can aid in the comprehension of a mathematical concept (Choi-Koh, 2003; Manouchehri, 2004; Touval, 1997). Studying one representation at a time and changing a single parameter or variable can help generalise over a class of outputs, such as a class of functions (Choi-Koh, 2003; Manouchehri, 2004) or a class of solutions to differential equations (Zeynivandnezhad & Bates, 2018). The affordances of high-speed calculations are even clearer when using spreadsheets. Spreadsheets offer an environment for exploring patterns of algebraic expressions, allowing students to generalise and explore conjectures of generalisations. In this way, the spreadsheet is used as a foundation for generalisation (Calder, 2012). Another affordance of a spreadsheet is to calculate different values of a mathematical expression based on a general formula. da Silva and colleagues' (2021) subjects generalised the formula from explorations in GeoGebra. In general, since CAS often include a syntax similar (yet, still different) to the generalised mathematical symbolic expression, mathematics learners often need to be able to recognise and formulate mathematically generalised symbolic expressions when using CAS and similar digital mathematics educational tools (e.g. da Silva et al., 2021; Zeynivandnezhad & Bates, 2018).

In the included literature focusing on DGE, dragging, using sliders and measuring are found to be the most affordable for generalisation (da Silva et al., 2021; Fonseca & Franchi, 2016; Khalil et al., 2017; Khalil et al., 2019; Leung, 2008; Sherman & Cayton, 2015; Turgut, 2019; Yao & Manouchehri, 2019). One of the main contributions of dragging when working with generalisation is the affordances of keeping some elements fixed while using variation to seek patterns and generalise over these through exploration (Fonseca & Franchi, 2016; Leung, 2008; Sherman & Cayton, 2015; Turgut, 2019). Based on Marton and colleagues' (2004) patterns of variation, Leung (2008) classifies types of dragging with different purposes, which are associated with different steps of generalising towards a robust construction or concept. In general, the literature features dynamic examinations of single cases facilitating an extension to another context, upon which a class of examples can be generalised to a more formalised mathematical concept. Yao and Manouchehri (2004) distinguish between different forms of generalisations – from one context to another, from examples to a class of examples or a formalised concept – and these generalisations are shaped by how the students use the tool and for what purpose.

Mathematical concepts' varying scope in different contexts

Six studies are categorised as featuring the scope of concept aspect ($N_{DGE} = 5$; $N_{CAS} = 0$; $N_S = 1$; $N_{GC} = 1$). Five studies involves DGE, one of which includes spreadsheets. One focuses on the graphic calculator, illustrating how the solutions using this tool can function as stepping stones for further investigation and extensions of a given problem (Touval, 1997). The presented results illustrate how DGE and graphic representations can aid in the process of relating to a concept's varying scope.

Using DGE in explorative ways to problematise mathematical problems and tasks can include extending a problem to other domains, settings or situations. When studying geometric constructions, DGE can assist in questioning the conditions and properties of certain geometric figures (Reyes-Rodriguez et al., 2017; Santos-Trigo & Reyes-Rodriguez, 2016). In this way, the ability to relate to the scope of a given geometric concept is included in the question–answer aspect of the MTC. Similarly, explorations of a problem in R^2, using DGE and spreadsheets, can lead to further explorations and extensions of the problem in R^3 (da Silva et al., 2021).

The affordances of keeping some elements fixed and varying others, as reported for the generalisation–abstraction aspect, also apply to the scope of concept aspect. These features of DGE make it possible to investigate a well-known concept, such as ”function” or ”mapping” in new contexts, as for instance linear algebra (Turgut, 2019) or geometric transformations (Yao & Manouchehri, 2019).

Mathematical statements and their logical structures

As mentioned, all included studies implicitly touch upon the ability to distinguish between the types and roles of mathematical statements as part of the other aspects of the MTC. For instance, many studies include elements of conjecturing and posing mathematical statements (e.g. Calder, 2012; Yao & Manouchehri, 2019; Zeynivandnezhad & Bates, 2018). Furthermore, when studying different conditions and requirements for constructing certain geometric figures, the participants need to focus on necessary and sufficient conditions, and thus, different kinds of claims. In this regard, DGE show potential, as students can construct and explore different dynamic models that can emphasise some of the properties of a certain construction (Leung, 2008; Reyes-Rodriguez et al., 2017; Santos-Trigo, 2004; Santos-Trigo & Reyes-Rodriguez, 2016). This is also the case when formulating conjectures of generalisations (Calder, 2012; Fonseca & Franchi, 2016), where generalisations are formulated as more or less formal mathematical statements (Yao & Manouchehri, 2019). Hence, students need to engage in the navigation of the logical connectives and quantifiers, such as implications, universal claims and existence claims.

This aspect is identified indirectly in the included literature, which may be due to its related activities. Formulating conjectures and working with mathematical statements and their logical structures may be more related to mathematical reasoning (e.g. Selden & Selden, 1995) in mathematics education research than to the general notion of mathematical thinking.

Discussion

The analyses of the reviewed literature illustrate possible ways of using DGE and CAS for the different aspects of the MTC. The findings of the review, naturally, reflect the review methodology. Restricting the search to literature on ”mathematical thinking” and ”thinking mathematically”, the findings are pertinent only to the use of DGE and CAS for activities that the work conceptualised as mathematical thinking. This method most likely excluded relevant literature from the beginning. For instance, papers involving the mathematical statements aspect could have been excluded, since mathematical statements are not related as explicitly to the general notion of mathematical thinking as some of the other aspects, such as the generalisation–abstraction aspect. Using keywords such as ”problem-solving”, ”problem-posing” and ”reasoning” could have indicated further studies for inclusion and illustrated other ways of using DGE and CAS to enhance the MTC aspects. However, since these keywords relate to problem handling and reasoning competency rather than the MTC, I found that using these confused the scope of the specified focus on the MTC. Despite the restricted search, the included literature still proposes possible ways to use DGE and CAS to support students' activities related to the MTC aspects and to keep the focus on the aspects of the MTC without confusing them with aspects of the other competencies.

Using DGE and CAS for activities of the mathematical thinking competency aspects

Regarding the use of DGE and CAS, the literature review indicates that DGE and CAS can have different affordances for student activities associated with the MTC. Spreadsheets and graphic calculators have some affordances that are similar to those offered by DGE and some similar to CAS. Therefore, with DGE and CAS being more distinct, their potential will be the focus of the discussion.

The included literature shows that CAS can provide multiple examples, which can lead mathematics learners to question and generalise problems, results and concepts. Using CAS as part of problem-solving and reasoning calls for activities of the MTC, such as the question–answer and the generalisation–abstraction aspects, enabling students to choose appropriate tools and formulas and to anticipate the feedback. In this way, CAS seems to play a more active role in problem-solving and reasoning activities prior to or after the reflections and activities of the MTC. In contrast, the included literature shows potential of DGE and their dynamic features, such as dragging to explore problems, results, concepts, relations and generalisations that interplay with the aspects of the MTC. For instance, the studies identified as involving the scope of concept aspect of the MTC mainly report on the use of DGE. This indicates that particular DGE have potential to aid in investigations of mathematical concepts in new contexts and domains. With digital technologies, students can handle geometric and graphic representation as they are real objects, which has some epistemic pitfalls (Duval, 2017). Yet, due to dynamic and easily repeated representations, inquiries using DGE can simulate experiments, making mathematical inquiry more accessible and concrete for the students.

Three aspects of the MTC were identified in the literature, namely the question–answer, the generalisation–abstraction and the scope of concept aspects. Respectively to these aspects, the analyses of the presented activities involving DGE and CAS indicate three ways that DGE and CAS can support students' learning activities related to aspects of the MTC.

1. For the question–answer aspect, mathematical problems can be studied using DGE or CAS before the actual problem-solving process. DGE and CAS can provide different constructions or examples of the same phenomenon, which can be compared and contrasted for the study of when a certain concept or phenomenon is represented and when it is not (either dynamically with DGE or by a one-to-one comparison with CAS). Thus, students can be encouraged to question the similarities and differences between the examples and to seek answers related to single cases, generalisations and implications.

2. For the generalisation–abstraction aspect, DGE can provide examples where the invariant and defining properties are preserved under dragging (Leung, 2008). Similarly, CAS can provide multiple representations illustrating invariant properties of a certain concept which structures, the students can investigate and thereby generalise and abstract.

3 For the scope of concept aspect, DGE can provide new representations of well-known problems or concepts in new settings. Students can investigate and explore these based on their previous knowledge and thereby extend the concept, the relation or the given problem to new domains and contexts. Utilising the affordances of multiple representational views, CAS may hold some of the same potential.

The mathematical statements aspect of the MTC is indirectly identified in the included studies in connection with the other aspects. By questioning problems, results and concepts, by studying special cases and by generalising and formulating conjectures, students can work with different kinds of mathematical statements and their involved claims and logical structures. This indicates that DGE and CAS, with their immediate feedback and multiple representations, also hold potential for students to deal with and relate to the different mathematical claims being involved in definitions and conjectures.

Simply applying DGE and CAS in the identified ways does not imply that students will engage in MTC-related activities. Niss (2016) argues that no digital tool or software is good or bad in itself. Teachers, learning environments and the formulation and implementation of tasks, as well as the role and purpose of the digital tool, are important when encouraging students to wonder, pose questions, share ideas and engage in mathematical inquiry and its particular modes of thinking. Despite their significance, the exact task design and the teachers' orchestration are not under the scope of this review.

Conclusion

The purpose of this review is to identify literature on mathematical thinking, focusing on the use of DGE and CAS in relation to aspects of the MTC of the KOM framework. The MTC covers different activities associated with engaging in mathematical inquiry, which in this article is divided into four aspects: the question–answer, scope of concept, mathematical statements and generalisation–abstraction aspects. The results of the review indicate that activities related to the MTC may be delicate situations related to reasoning, problem-solving and problem-posing.

This paper aims to give the aspects of the MTC more attention in teaching and learning situations involving DGE and CAS by using a restricted search strategy focusing on the general notion of mathematical thinking and the use of DGE and CAS. In the literature explicitly addressing the interplay between mathematical thinking and the use of DGE and CAS, three of the four MTC aspects were identified, indicating three ways in which DGE and CAS can be tools for activities to enhance MTC, with one for the question–answer, one for the generalisation–abstraction and one for the scope of concept aspects. The mathematical statements aspect was implicit identified due to its connection with the other aspects. The lack of consideration in isolation from the other aspects can be due to mathematical statements not being explicitly related to mathematical thinking, such as generalisation and abstraction are. However, since the mathematical statements aspect is part of the others, the review also indicate potential of DGE and CAS to support this aspect.

In the identified studies in the review, DGE show to be involved in the actual activities of the MTC. Studies focusing on CAS are more involved with the activities of problem-solving and reasoning as activities prior to or after aspects of the MTC. Nevertheless, the results show that CAS also holds potential for supporting the question–answer and the generalisation–abstraction aspects of the MTC. This leaves the potential for further investigations on the use of CAS to support aspects of the MTC. Of course, as is the general case when using digital technologies, simply providing the tools for students is not enough to enhance MTC-related activities (e.g. Guin & Trouche, 1998). For students to engage in these delicate situations, specific tasks for activities related to the MTC need to be integrated as part of task sequences and lessons before and after students dive into the other activities of problem-solving and reasoning.

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