Didactical usefulness of interactive mathematical maps – designing activities supporting prospective teachers' learning

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Prospective teachers often have difficulty in linking school mathematics and university course content, which manifests itself as a lack of their understanding and the significance of university course content. This double discontinuity is experienced by future teachers in their transition from high school pupil to university student and then from university student to their school teaching career. Thus, it is necessary to improve university teaching and teacher education to try to bridge these ″gaps". Using the educational context of a geometry course in the teacher education of upper secondary teachers, we explore the technical implementation and usefulness of the components of *interactive mathematical maps*. Such maps comprise a supplementing didactical tool that shows the interrelations between mathematical discoveries and the development of particular mathematical content – starting from an initial historical problem situated in time. The research findings showed the map in its current format to be perceived as useful and mostly easy to use. Further, the map seemed to promote both a process-oriented and an application-oriented approach as well as favourable beliefs, such as mathematics being an emerging science promoting a view of *doing mathematics*, in which an open error culture can be established.

Addressing the double discontinuity via defragmentation

In some countries, such as Sweden and Germany, prospective teachers experience a long training period to ensure high quality in their future teaching. However, many studies have indicated the fact that prospective teachers often have difficulty in linking school and university content, which manifests itself in them lacking understanding and the

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significance of university content (e.g. Hefendehl-Hebeker, 2013; Pinto & Cooper, 2022; Winsløw & Grønbæk, 2014). Thus, it is necessary to improve university teaching and teacher education by trying to bridge these ″gaps″.

More than a century ago, Klein (1908/1932) pointed out that it is by no means a natural thing for future teachers to be able to link school and university content in a meaningful way. He described this as a ″double discontinuity″ (Klein, 1908/1932), a double phenomenon arising from the transition of a high school pupil to a university student and then from a university student to a school teacher. Describing the first discontinuity, Klein wrote: ″[When a student is] confronted with problems, which do not remind him, in any particular way, of the things with which he had been concerned at school. Naturally, he forgets all these things quickly and thoroughly″ (Klein, 1924/2016, p. 1). Thus, the underlying reason for this is that the different ways in which mathematics appears at school and at university lead to students being unable to relate them to each other. A lack of connection between the content of school and university mathematics ultimately results in the design of poor teaching. Winsløw and Grønbæk (2014) showed in their study that Klein's second discontinuity is still a problem for mathematics teachers, who often fail to transfer their ″academic knowledge gained at university to relevant knowledge for a teacher″ (Winsløw and Grønbæk, 2014, p. 2), especially when it comes to autonomous work (pp. 7–15). The double discontinuity phenomenon implies that teachers may not rely on their academic mathematics knowledge gained as a result of their university studies but instead base their school teaching on their own pre-university experiences (Bauer & Partheil, 2009; Hefendehl-Hebeker, 2013). Consequently, the content of mathematics teaching becomes fragmented, and students' have fewer opportunities to notice connections and develop conceptual understanding (Winsløw & Grønbæk, 2014). Thus, research has suggested that the phenomenon of double discontinuity is unfortunately still very much alive, regardless of changes over time in the teaching process both of mathematics in schools and in teacher education in universities. It still seems that prospective teachers strongly believe that the topics in university mathematics courses do not meet the demands of their later profession in schools (Ableitinger et al., 2013; Hefendehl-Hebeker, 2013; Isaev & Eichler, 2018).

Winsløw and Grønbæk (2014) classified Klein's double discontinuity into the following three dimensions, which are not independent: *the institutional context* (of university v school), *how the subject's role differs between institutions* (a student at university or school v a teacher of school mathematics) and *the difference in mathematical content* (elementary v scientific). Klein mainly focused on the third dimension, proposing new approaches in the teaching of mathematics at university level that aim to bridge these gaps. He claimed that "university instruction [should consider] the needs of the school teacher, [...] help them see the mutual connection between problems in the various fields [...] and more explicitly to emphasise the relation of these problems to those of school mathematics" (Klein, 1908/1932, p. 1–2). To overcome Klein's double discontinuity, students are required to notice connections between the "axiomatic-formal" world of university mathematics and the "perceptual-symbolic" or "conceptual-embodied" world of school mathematics (cf. Tall, 2008). Unfortunately, students are not able to find those autonomously or incidentally (Winsløw & Grønbæk, 2014). The lack of perception of intra-mathematical connections leads not only to the fragmentation of school and university content but also to that of teacher education as a whole, since the different mathematics courses, as well as mathematics and mathematics education, are likely to be perceived as isolated from each other.

As a way to address this, we develop and evaluate a learning tool, *interactive mathematical maps* (as introduced in Brandl, 2009), that specifically reveals connections between school and university mathematics. The basic idea is that the double discontinuity can be overcome by the defragmentation – a thematically meaningful arrangement – of knowledge. The digital interactive mathematical map has the purpose of enabling "meaningful learning" in the sense of Ausubel (1963), who declared that learning (other than rote learning) is only meaningful when connected to both past learnings and many different contexts. By giving the learners the opportunity to retrace the formation of mathematical ideas in conjunction with the presentation of similar topics, understanding is deepened, and the transfer of learning is ultimately facilitated.

The underlying epistemological view on mathematics focuses on a fallibilist perspective (Ernest, 2014), with its emphasis on social processes and the influence of the human component. With respect to data implementation, every mathematical topic and idea is always connected to a person or protagonist, namely the specific mathematician and his or her biography. The use of narrative didactics then allows for those elements to be an historical anchor in the teaching and learning process (see below). The desired "dynamic" beliefs structure (Felbrich et al., 2012, Felbrich et al., 2008) also represents that underlying epistemological view (see below).

Using the educational context of a geometry course as part of teacher education for upper secondary teachers (*ämneslärare*), on the one hand, we explore the technical implementation and usefulness or ease of use of the components of the mathematical maps, and on the other hand, we ask the following question: *does the use of mathematical maps in a geometry course help to counteract the second discontinuity with an emphasis on promoting favourable beliefs related to the nature of mathematics*?

The first part of the research question is answered by using the technology acceptance model (TAM; see figure 4) of Davis (1985) to analyse students' answers to questions on the perceived usefulness and ease of use of the technology (the tool) in weekly assignments. The second part of the research question is answered to a certain extent by the results of qualitative content analysis regarding different categories of beliefs structures (according to Felbrich et al., 2008) from free-text answers and students' own produced texts based on the content of the map (see below). In sum we discuss the perspective of the beliefs in relation to the use of the interactive mathematical map.

Interactive mathematical maps

The concept of mathematical maps was introduced by Brandl (2009) as a supplementing didactical tool that shows interrelations between mathematical discoveries (our horizontal dimension) alongside the development of a mathematical topic, starting from an initial historical problem in time (vertical dimension) (see figure 1). The interactive three-dimensional mathematical map is based on a constructivist view of learning and intended to "offer the student an optimal solution for establishing successful learning processes" (Brandl, 2009, p. 106) by integrating the historical origin of mathematical concepts with the interdependencies between them. The mathematical map should combine these two characteristics in one three-dimensional representation as a kind of graph or tree. One dimension represents time and the other two represent inner-mathematical dependencies.

Since this article focuses on the development and research process of mathematical maps used as a didactical tool in a geometry course as part

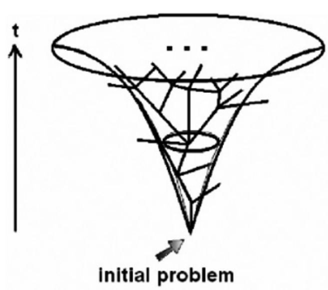

Figure 1. *Initial idea for the design of the didactical tool (Brandl, 2009)*

of teacher education, only central components that are important for the later evaluation are outlined here. More detailed information about the construction of interactive mathematical maps and the previous research studies can be found in Przybilla et al. (2021) and Przybilla et al. (2022).

In the map, nodes in space represent mathematical topics and ideas that are linked to descriptions and further free content from the world wide web (see figure 2). The edges symbolise historical developments that emphasise mathematics as an *emerging science* that is "intuitive and genetic; that is, the entire structure is gradually erected on the basis of familiar, concrete things" (Klein, 1924/2016, p. 9). To give an additional overview of the similarities between subjects, the remaining two dimensions are defined based on thematic relatedness (more details are given in Przybilla et al., 2021).

As described in Przybilla et al. (2022), when clicking on a node, all content of the node is displayed on the right side of the screen, and the various functionalities of the map can be applied via a bar at the top of the screen (see figure 2). In the *horizontal cut* function, all nodes are projected into a plane so that statements can be made about the similarity of content via the Euclidean distance. In this way, inner-mathematical relationships should become clear. The nodes can be filtered by topic so that similarities become even more apparent.

In the *vertical cut* function, any node can be selected, for which the historical genesis is then represented as a directed graph in two dimensions (see figure 3). The user can follow the graph, and therefore the development, directly by using the preview on the right side of the screen.

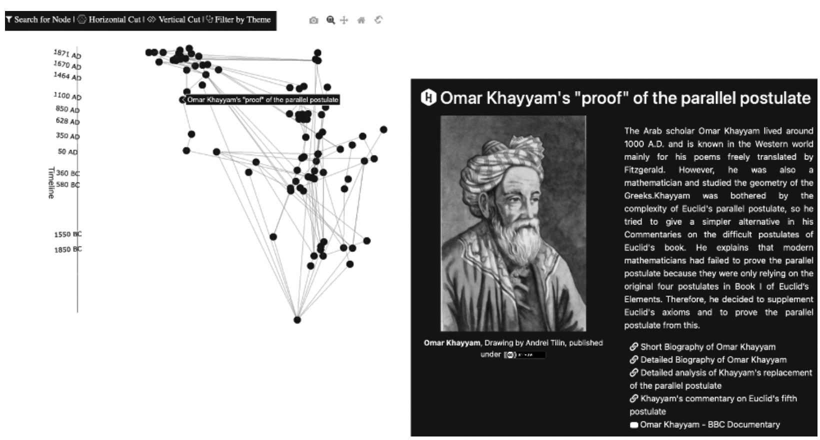

Figure 2. *Screenshot of the three-dimensional mathematical map for geometry (Status: 14.09.2022)*

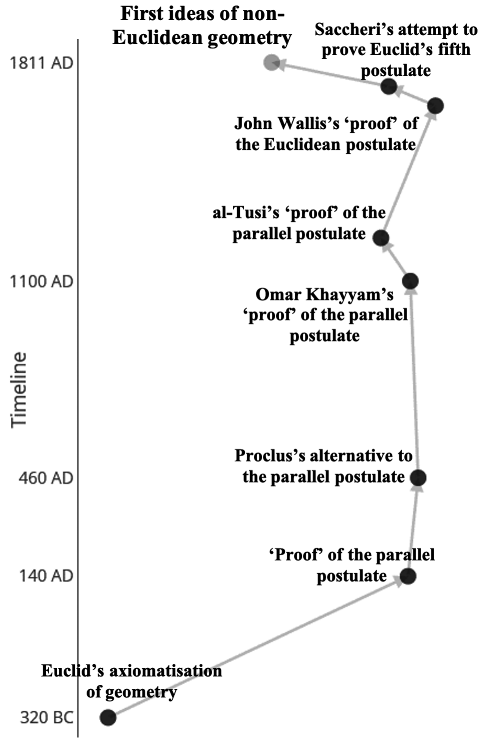

Figure 3. *Vertical cut for the "first ideas on non-euclidean geometry" node filtered by Date and Theme (Labels shifted, Status: 14.09.2022)*

In addition to providing a better understanding of the mathematical content, the main goal of the vertical cut is to promote an accurate picture of mathematics as an *emerging science*; mistakes and misunderstandings can and probably will occur when *doing* or *creating* mathematics. We illustrate this with the well-known example concerning the centuries-long discussion about Euclid's parallel postulate (see figure 3), which was also explored in the aforementioned geometry course. The example illustrates the connection of school mathematics (e.g. Euclidean geometry) with university mathematics (e.g. non-Euclidean geometry).

The underlying atomistic definition of knowledge is based on the two central assumptions of the "de-composability" and "de-contextualisation" of learning content (Resnick & Resnick, 1992). This fundamentally contradicts the networking/connecting idea of modern (mathematics) didactics, which emphasises the networking/connecting character in terms of both the content and the teaching process (Brandl, 2016). With respect to the content, a visual connection of the topic is given by the *interactive mathematical map* virtual/digital tool. With respect to the teaching process, a connection of the topic (e.g. amongst others) is given by the methodological use of narrative didactics. Authentic excerpts from the historical context of the history of mathematics could help to make the vitality of science, which shows itself in research, discovery and creativity, visible again (Klassen, 2006). Kubli (1999, 2002) reported that students react in a much more positive way to historical material if it is prepared and used in a narrative form. The aim of also using narrative elements in the teaching process is to get the learners to engage emotionally with the narrative and therefore also with the mathematical content. The affective context is always involved (see Egan, 1989a, 1989b) and must be considered when constructing narrative elements. Bruner (1986) had already contrasted the logical-discursive argumentation (”logico-scientific mode”) with the ”narrative mode” of thinking. While Norris et al. (2005) found helpful transformation aids from the logico-scientific mode to the narrative mode, Klassen (2006), with his story-driven contextual approach, provided a practicable concept for teaching adapted in this course in terms of the mathematical map. By embedding the topic in a wider context in any way (e.g. related content areas, historical events, narrative elements or stimulated affects), ”anchor points” are already evident, and these allow the topic to be more firmly integrated into the learning context and eliminate its isolation. In this sense, networking/connecting (e.g. in the form of a rich learning environment using the mathematical maps tool accompanied by narrative elements) takes on the role of a supporting framework to promote the individual learning process. It thus corresponds to a ”scaffolding process”, as described by Wood et al. (1976, p. 90). The stabilising anchoring/foundation of the topic in these contexts can now lead to a better understanding of the topic. A more detailed summary and description of these approaches, as well as an exemplary extension to pictorial narration, can be found in Brandl (2017), on which part of the intervention in this study was based to illustrate the application of narrative didactics for students.

Foundation for the technical evaluation

As described in Przybilla et al. (2022), we adopted the TAM of Davis (1985) (see figure 4), which is widely used in the community and well supported both theoretically and empirically (cf. Scherer et al., 2019), as the foundation for the technical evaluation of the digital tool.

According to the model, a potential user’s overall attitude to using a given system is a major determinant of whether or not he or she will actually use it. Attitude towards use is a function of two major beliefs, namely perceived usefulness and perceived ease of use. Perceived ease of use has a causal effect on perceived usefulness. Design features directly influence perceived usefulness and perceived ease of use (Davis, 1985, p. 24); therefore, these perceptions determine whether or not a digital tool is used. This in turn depends crucially on the properties of the design features. In our case, we consider as design features the individual components and functionalities of the map (see figure 4).

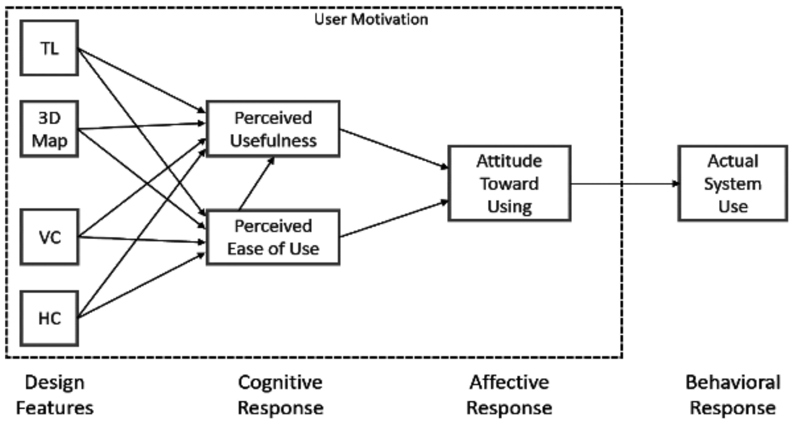

Figure 4. *Illustration of the technology acceptance model for evaluating the mathematical maps (Davis, 1985)*

Analysis of beliefs

We use the students' free-text answers, general evaluation (GE) of the tool and texts produced from weekly assignments (as an application of the content in the mathematical map) to derive hypotheses about the didactical usefulness of mathematical maps by qualitative content analysis.

We analyse the students' didactic beliefs concerning the nature of mathematics, which "are a crucial part of the professional competence of mathematics teachers" (Felbrich et al., 2008, p. 763), which the use of the map seemed to promote.

Beliefs can be defined in different ways. While applying the structure of practicing teachers' beliefs on mathematics, as described in Felbrich et al. (2008), we stick to the same "broad definition" (p. 763) describing beliefs in an open meaning as "psychologically held understandings, premises, or propositions about the world that are felt to be true" (Richardson, 1996, p. 103, cited in Felbrich et al., 2008, p. 763). According to Felbrich et al. (2008) we also "assume that beliefs comprise of affective, motivational and cognitive aspects of knowledge" (p. 763).

In this context, a distinction is made between the following principal orientations, which also provide the categories for the content analysis: the formalism-related orientation, the scheme-related orientation, the process-related orientation and the application-related orientation (Felbrich, 2008, p. 764, also for detailed definitions). The belief structure is one-dimensional (see figure 5) and has two poles: a dynamic perspective, indicated by *process and application*, and a static one, indicated by *formalism and scheme*. The poles are seen as "mutually exclusive and antagonistic, in the sense that a person favours either a dynamic or a static view on mathematics" (Felbrich, 2008, p. 764). The static perspective gives a non-accurate picture of mathematics, which is *unfavourable* for teachers and learners (Felbrich, 2008). The dynamic understanding of mathematics emphasises that the subject as an *emerging science* in which failures belong. It promotes an accurate view of *doing mathematics*, in which an open error culture can be established. These are *favourable beliefs* related to the nature of mathematics (Felbrich, 2008). By analysing students' didactical beliefs concerning the nature of mathematics after using the interactive mathematical map, we try to find out if *favourable beliefs* are promoted, encouraging a modelling perspective and application of mathematical knowledge in a meaningful context, helping to counteract the double, and especially the second, discontinuity.

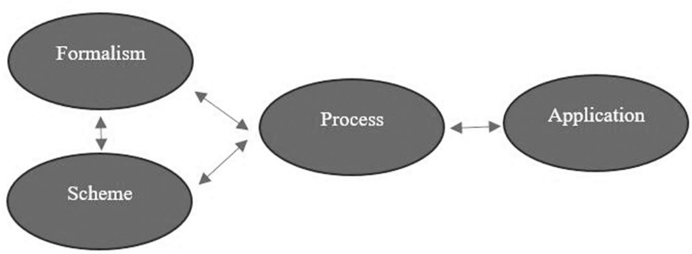

Figure 5. *Structure of beliefs on mathematics of practicing teachers, as described in Felbrich et al. (2008)*

Design of the study

In the current study, a mathematical map was used as part of a geometry course (7.5 ECTS credits) at Karlstad University. The course was anchored in the second semester of the teacher training programme (300 ECTS credits) and could also be attended by teachers who wished to receive further mathematical training. The components of the course had been designed cooperatively prior to the study by mathematics teachers and researchers of mathematics education from Karlstad University and the

University of Passau. The content-related goal of the course was to cover Euclidean geometry and to introduce key concepts from non-Euclidean geometry. In addition, the module catalogue required the development of competencies for the evaluation of digital learning tools, saying: "After completing the course, the student must be able to use and analyse the use of digital learning tools (e.g. dynamic geometry programs) and account for students' thinking in geometry."

Participants

In the semester under consideration, 44 of a total of 64 students (27 teachers and 17 prospective teachers) actively participated in the study. The examinations included in the study were mandatory, but participation in the study was not.

Intervention

The geometry course's weekly lectures were supplemented by content and weekly work assignments[1] dealing with the digital didactical tool so that an evaluation of the individual components of the mathematical map could take place.

The weekly work assignments were arranged chronologically as follows: (a) timeline (TL); (b) narrative didactics (ND), as an area of application of the content of the mathematical map; (c) the three-dimensional mathematical map (3DM); (d) the vertical cut (VC); (e) the horizontal cut (HC); (f) discussion on the parallel postulates (PP) as an area of application of the content in the mathematical map; and (g) a general evaluation of the didactical tool (GE).

For each of the components of the map (TL, 3DM, VC, HC), a short explanatory video of no more than 10 minutes was created. Afterwards, the participants were required to try out the respective functionalities themselves in a work assignment. The assignment concluded with a technical and content evaluation of the individual components of the map, organised in a quiz with both single-choice and free-text questions.

To support assignments 2 (ND) and 6 (PP), a 2-hour lecture was given to introduce the students to narrative didactics by using topics on non-Euclidean geometry.

Procedure

In the present study, we evaluate the usefulness (see TAM, figure 4) of the components of the mathematical map as a tool to overcome the problem of Klein's double discontinuity. Alongside this, we explore how to improve the map's technical elements so that a user will find it easier to explore. The main data for our study consisted of students' responses to the weekly assignments and their reflections on the use of the map.

All the assignments were focused on content exploration and had a cognitive aspect. However, the students were also asked to reflect during their work on the use of the map and all its functionalities, adding a new technical aspect as well as a didactical one, to move the focus beyond the cognitive. Table 1 provides an overview over technical and content evaluation of the work assignments.

Table 1. *Technical and content evaluation of the work assignments*

Work assignment	Student activity	Technical and content evaluation
Timeline	The students familiarise themselves with how the timeline works by selecting all the content from the map that was developed by the Babylonians, Egyptians, Greeks, Indians and Arabs (from the beginning to 1000 AD). They are meant to choose only the nodes encountered in school and create a dynamic web page with them.	Single-choice and free-text questions regarding overall functions, such as "The time line is simple to use", "The selection function is useful", "The fact that videos are bookmarked is useful" and "The preview function helps me to navigate through the map", and regarding perceived usefulness (TAM), such as "I can imagine myself in the future, when learning new content at the university, informing myself about its origins and development".
Narrative didactics as an area of application of the content in the mathematical map	The students are asked to formulate a short historical-orientated narrative motivation for a school topic of their own choice with the help of the information provided in the timeline of the interactive mathematical map.	Single-choice and free-text questions regarding perceived usefulness (TAM), such as "I have experience with the historical-genetic principle", "I think that knowledge of historical lines of development of mathematical concepts will help me in my future profession as a teacher" and "I experience the teaching of mathematics at the university as orientated on the historical-genetic principle".
Three-dimensional mathematical map	The students are supposed to familiarise themselves with how to use the map by following the development line from the starting point of "Real objects" to the "Closed formula for the circle constant pi" node.	Single-choice and free-text questions, such as questions of overall functions related to perceived ease of use (TAM).
Vertical cut	The students are instructed to look over the vertical cut of the "Exponential, sine and cosine series" node and sort all nodes in the development tree related to their affiliation to school or university mathematics.	Single-choice and free-text questions specific to the function, such as "I see the vertical cut functionality as a useful and clear way to show historical developments", related to perceived ease of use (TAM).
Horizontal cut	The students are asked to use the horizontal cut function to get an overview of the map's thematic areas and search for thematic clusters. In addition, they are supposed to use the filter function for each cluster to no longer display interfering nodes	Single-choice and free text questions specific to the function, such as "I see the horizontal cut functionality as a useful and clear way to show thematical connections", related to perceived ease of use (TAM).
The discussion around the parallel postulates as an area of application of the contents in the mathematical map	In the parallel postulates assignment, the students are asked to use the map to describe how non-Euclidean geometry evolved from Euclidean geometry, which dominated for centuries.	This was not explicitly technically evaluated. However, single-choice and free-text questions related to the intervention, such as "I think that knowledge of the historical lines of development of mathematical concepts will help me in my future profession as a teacher" and "I experience the teaching of mathematics at the university as oriented on the historical-genetic principle", are related to this assignment and give information regarding students' earlier experiences
General evaluation of the didactical tool	The students respond to an evaluation form.	Single-choice and free-text questions regarding both perceived ease of use and perceived usefulness (TAM).

The weekly assignments, which can be accessed via the link included in the footnote, were constructed so that the students could reflect on their work and the tool they had used. In the ND assignment they were asked to formulate a short historical-orientated narrative motivation for a school topic of their own choice with the help of the information provided in the timeline of the digital interactive mathematical map. In the PP assignment, the students were asked to use the map to describe how non-Euclidean geometry had evolved from Euclidean geometry, which dominated for centuries.

At the end of the course, an overall evaluation was carried out, in which learning gains from the use of the map and expected applicability for teaching with the map in schools were to be assessed.

All weekly assignments, except for the final two, concluded with a brief technical and content evaluation of the individual components examined that was organised into a quiz with both single-choice questions, which were laid out using a five-point Likert scale, and free-text questions. In addition, the teachers and prospective teachers had the opportunity to use text fields to provide feedback and suggestions on how the learning tool should be improved. When analysing the data, the participants were already anonymised, so we could not track if a student was a pre- or in-service teacher.

Findings

Ease of use and usefulness of the digital tool

The first part of the study focused on the technical evaluation of the digital tool interactive mathematical map with the purpose of both improving the tool and getting an overview of its perceived usefulness and ease of use according to TAM (Davis, 1985). For this purpose, we used quizzes after the assignments that had both single-choice questions, which were laid out using a five-point Likert scale, and free-text questions concerning the functionalities of the map. The purpose of the quizzes was to give the students an opportunity to reflect on their work and the tool they had used. Overall, few of the students (6 %) agreed or strongly agreed that they had previously experienced the historical-genetic principle (i.e., before the course). Of the students, 22 % agreed or strongly agreed that they had experienced the teaching of mathematics at the university as oriented on the historical-genetic principle. In table 2, we show the questions that we asked the students following their experiences with the timeline, the three-dimensional map, the vertical cut and the horizontal cut. Overall, the students largely agreed or strongly

Table 2. *Evaluation of the timeline, the three-dimensional map, the vertical cut and the horizontal cut*

Question	Strongly agree	Agree	Neither agree nor disagree	Disagree	Strongly disagree
Perceived ease of use					
The timeline is simple to use.	31 %	54 %	13 %	0 %	2 %
The selection function is useful.	37 %	40 %	17 %	4 %	2 %
The fact that videos are bookmarked is useful.	69 %	15 %	26 %	0 %	0 %
The preview function helps me to navigate through the map.	28 %	46 %	19 %	7 %	0 %
Perceived usefulness for students					
I use the search function to find a node in the map that interest me.	48 %	37 %	11 %	2 %	2 %
I use the filter function to find nodes that interests me.	37 %	48 %	9 %	4 %	2 %
The "filter date" functionality helps me get deeper insights about the development of mathema-tics and the circumstances of this development.	33 %	60 %	7 %	0 %	0 %
I can imagine myself in the future, when learning new content at university, informing myself about its origins and development.	22 %	53 %	22 %	2 %	0 %
Perceived usefulness as (a becoming) teacher					
I see the vertical cut functionality as a useful and clear way to show historical developments.	51 %	40 %	9 %	0 %	0 %
I see the horizontal cut functionality as a useful and clear way to show thematical connections.	43 %	50 %	5 %	3 %	0 %
I think that knowledge of the historical lines of development of mathematical concepts will help me in my future profession as a teacher.	42 %	40 %	18 %	0 %	0 %

agreed with the perceived usefulness of the tool both as students and as (becoming) teachers.

Over 80 % of the students found the timeline to be useful for presenting historical events in a clear and structured way. A content table or a search function within the timeline was requested several times to provide a clear overview of the included content. This could increase the given percentage. In the three-dimensional map, 89 % of the students found both the search and the filter functions to be useful. The preview within the map was also perceived as useful by almost 80 %. A large portion of the remaining 20 % was bothered by a technical detail that has since been adjusted. While few students perceived the timeline as difficult to use, there was more difficulty with the 3D map, and a brief training session was necessary for the fluent handling it. This must be considered when using the map in another course. Almost all (95 %) of the participants agreed to the statement that "I see the vertical cut functionality as a useful and clear way to show historical developments". In this functionality, the students and teachers saw the greatest opportunity for the use of the didactical tool as being in schools. The horizontal cut was seen by 93 % as a useful and clear way to show the thematic connections of mathematical content.

The free-text answers on the functionalities of the map showed that the students had intensively dealt with the map and discussed its usefulness for teaching and learning processes in depth. Examples of this included a request for a user manual for pupils, several design suggestions and desire for an improvement of the zoom function in the horizontal cut and general improvement of the functionality for touch devices (especially of the 3D map).

In the quizzes, we ask about the perceived ease of use and perceived usefulness of the individual components of the map, reflecting the cognitive response. In the next step, we look at attitudes to using the tool reflecting the affective response and at actual system use reflecting the behavioural response (as in figure 4). Therefore, we are able, based on TAM (Davies, 1985) to draw conclusions about the ease of use and usefulness of the entire digital tool.

Favourable beliefs concerning the nature of mathematics

To counteract the second discontinuity, we analysed students' didactic beliefs concerning the nature of mathematics after using the interactive mathematical map. For this, we used the free-text answers from quizzes and general evaluation. The students explained how the map connected the university mathematics course and the subject taught in school, thereby opening up the didactical transfer of academic knowledge to relevant knowledge for teaching (cf. Winsløw and Grønbæk, 2014).

In the free-text answers[2], the students reflected on the map's perceived usefulness for student learning and possible usage during teaching. The students express a *dynamic understanding*, emphasising mathematics as an *emerging science*. They connected it to future teaching, promoting a view of *doing mathematics*, whereby an open error culture can be established.

> To learn the history of mathematics as a teacher is very important. First, in my own learning process, it helps me understand that *mathematics is a subject that has evolved over a long time and in close proximity to practical problems*. That gives an extra perspective on the areas that we study – in this case geometry. Second, it also helps me

> in my teaching process. Many students see mathematics only as a subject taught in school, and by referring to the context of mathematical history, *I could help them understand that mathematics is more than that.* (Student 3)

> [The map] is of great benefit to *provide insight into the development of mathematics*. This also helps and gives *a better understanding of how to teach it* in school. Everything has a background. (Student 5)

The students showed awareness of the purpose of the map. They described how the map "shows how mathematics has evolved through time, in many parts of the world, by many mathematicians, either in parallel or through an evolution of mathematical findings" and that it displays a "holistic view of the evolution of mathematics by which it is possible to follow major development paths over time". The students' reflections indicated a *dynamic understanding*; the map seemed to make them open to viewing *mathematics as an emerging science* and reflecting upon how this may shape teaching and learning.

> I believe that letting students understand that mathematical *concepts are developed* from all over the old world (Europe, Asia and Africa) is important knowledge for both me and my future pupils. Another important fact for pupils to know is that *everything was not discovered at once but there have been developments*; one discovery leads to another, and *some concepts are dropped because of a new discovery.* (Student 24)

> The history of mathematics is cultural history and of equal importance to the history of, for example, literature or music. It is also crucial to know that mathematics has *developed over the years* and has *not been produced in the "parcels" we consume from textbook*s.
>
> (Student 28)

Even though many reflections on the map's *perceived usefulness for student learning* and *possible usage during the teaching process* were positive, some students expressed some doubts vis-á-vis their own learning process. This might occur if a student's prior belief system is challenged regarding what type of mathematics to engage in or how to study mathematics.

> I am not sure how much the interactive map helped me *in my own learning in this course when it came to the topics that were covered in the exam*. However, from a personal perspective, I learned a lot, since the interactive map *helped me in understanding the relationship between the different discoveries in mathematics.* (Student 21)

> I think that the map is a tool that works very well for some and not as well for others. I am a person who generally does not use online sources when trying to learn maths. Most often, t*he course literature and written notes from lectures are sufficient for me*. Therefore, I want to start by saying that I did not use the map as a learning tool to the extent that it could have been used. Nevertheless, I have to say that *when distinguishing the various geometries that have been created, the mathematical map was a great help*. Finding the source (proving the parallel postulate), the new branches that grew and became non-Euclidean geometry, spherical geometry, neutral geometry, etc., were much easier than they would have been if I had Googled them on my own. (Student 23)

In the analysis of the free text, we also detected signs of a more *scheme-orientated view* that may have prevented the digital tool (the map) from counteracting the second discontinuity.

> I would say that the hard part that required some work *was to interpret and understand all the knowledge that existed on the different nodes*. It required a bit more effort and was not as obvious, I would say. For me, it would require either a lot of time or a little more prior knowledge of some subjects for me to see connections and understand development more quickly. (Student 2)

> I must be honest and say that I found it more useful to learn geometry from our lectures and the books we used. What I did learn most from the map was the history of geometry. The map with the timeline made me realise what was discovered first, which gave a good understanding of how geometry has developed through the centuries. When I answered quiz 6, I really liked how you could filter by different themes. This *was not to learn geometry, but I could use it to understand how different axiom systems had developed*. (Student 1)

The students also reflected on the *usefulness vis-à-vis motivational aspects of learning* (both as a student and in relation to becoming teachers in schools) to enrich their views on mathematics.

> I got a better overview of when different mathematical events took place and felt that it became more interesting to be able to follow mathematics through this timeline. So, I would very much like to use this timeline in my teaching in the *hope of motivating students to see mathematics as something interesting that has developed* over a long period of time. (Student 19)

> I think it can help me associate maths with the real world. *Maths will then become less about random discovered truths and more about understandable processes of challenges and the overcoming of these challenges*. I think this will help some *students who struggle to find motivation for maths*. (Student 34)

The considerations about the usefulness of the map also included reflections on *the relations to the structure of mathematics* and students' opportunities to learn, thereby indicating a *dynamic understanding* of mathematics.

> For me, the most important thing was the opportunity to *get a clear view of the development of mathematics*. In the timeline, you can actually see how different events are connected to each other. (Student 7)

> I believe that this is the reason why mathematics is so difficult to learn for so many people, since everything is more or less connected, and *every student of mathematics needs to find the inner structure to get a grip*. This is a marvellous *tool for doing so*. (Student 13)

In the ND assignment, the students were asked to formulate a short historical-orientated narrative motivation for a school topic of their own choice with the help of the information provided in the timeline of the interactive mathematical map. The quality of the texts concerning the principles of narrative didactics was not a subject of analysis in this article. However, it is worth mentioning that 90 % of the students completed (according to elements of narrative didactics) the assignment successfully. Instead, we concentrate on students' use of the map to reach the goals of the assignment. In table 2, we show the questions given to the students following their experiences with the ND assignment.

Table 2. *Evaluation of narrative didactics*

Question	Strongly agree	Agree	Neither agree nor disagree	Disagree	Strongly disagree
The content in the timeline is helpful for writing a narrative motivation.	78 %	0 %	16 %	6 %	0 %
I have already been exposed to narrative elements in my university education.	22 %	33 %	34 %	11 %	0 %
I can imagine incorporating narrative elements into my future teaching.	56 %	39 %	5 %	0 %	0 %
I would like to see teaching at university spiced up with narrative elements.	11 %	56 %	22 %	11 %	0 %

Almost 80% of the students found the content in the map useful for writing a narrative motivation that could be used in their teaching, despite the fact that only half of the students had experienced such teaching before. Moreover, 95% of the students wished to incorporate narrative elements in their future teaching, and almost 70% wanted to experience this approach in university courses. The students' free-text comments also revealed the perceived usefulness of applying the content of the map to a practice-related assignment, since students found it ”rewarding [...] to be able to produce a text oneself!”. Discovering narrative didactics and its power as a didactical tool in mathematics teaching was another topic that was discussed. Thus, the map seemed to be a good source of inspiration for the students' own written texts on narrative motivation, and the assignment opened up new didactical opportunities for the prospective and in-service teachers.

In the PP assignment, the students were asked to use the map to describe how non-Euclidean geometry had evolved from Euclidean geometry, which had dominated for centuries. A total of 85% of the students completed the assignment successfully. We concentrated this time on the students' free-text comments regarding the usefulness of the map for this purely cognitive assignment. Most of the students found the information in the map relevant and easy to use for the assignment purpose and for their own understanding. One student affirmed it as follows.

> I did use the interactive map while writing the short article on non-Euclidean geometries. [...] It was very interesting and helpful (and fun) shifting through the nodes, reading about how non-Euclidean geometries developed. I could easily have written an article 10 times longer on the subject and enjoyed doing it. The interactive map was easy to use and gave a great overview of the history of mathematics.
>
> (Student 15)

To overcome Klein's double discontinuity, our data suggest that students gained a better understanding of the axiomatic-formal world of university mathematics. In connection with the PP assignment, students ”could use [the map] to understand how different axiom systems had developed”.

> My first trials with the tool gave me many good moments and interesting findings, but to be honest, it was not until the hands-on task to find the development from Euclidean to non-Euclidean geometry that I actually used the tool seriously. So my conclusion is that it was good for me when I had a specific task to find out more about.
>
> (Student 18)

The actual system use was tested in two of the weekly assignments (ND and PP), which were constructed as direct applications of the content of the mathematical map. The students expressed through their free-text comments ideas concerning the actual system use.

> I am a teacher and have already applied the interactive map in my teaching. It gave the students and me a completely different perspective on mathematics. I felt incredibly professional, even though I did not know much. To describe mathematics in ordinary words and to be able to search for connections in the interactive map is only limited by the imagination. When I teach different parts for the students, we will continue to use the map and put the mathematics in context. The students started listening to me in a completely different way. They asked questions, and I felt that mathematics had become available to them. I teach mathematics, science and technology in year 9. I have also used the map in physics, when we talked about the difference between Newton's and Einstein's theories. It was the best explanation ever, or so it felt, thanks to the interactive map. (Student 16)

The above example also shows that the use of the map supported the "didactical transfer of academic knowledge to relevant knowledge for teaching", helping to counteract the double discontinuity (cf. Winsløw & Grønbæk, 2014). As one student put it:

> [The] project including the map was something of an awakening and an eye opener. Studying the history of mathematics also gave me a reminder that mathematics is a human construction and that much of what we may take for granted today (analysis, spherical geometry, etc.) was "invented" not too long ago. It is interesting to reflect on and makes me think about the future and what the people then will take for granted. By studying the map, I can get a picture of and quickly see many of the important events in the history of mathematics. For me, understanding that mathematics is a human construction makes the legacy of the discoveries of the time and mathematics research feel much less distanced, which makes it easier to understand. (Student 29)

Discussion

With respect to our two-piece research intention, we summarise and discuss here the extent to which the use of mathematical maps in the geometry course helped to counteract the second discontinuity with an emphasis on promoting favourable beliefs related to the nature of mathematics and how the technical implementation and usefulness/ease of use of the components of the mathematical map were perceived.

From a didactic perspective, and concerning our research question, the evaluation results seem to indicate that the use of the map (within this course) contributed to *favourable beliefs* concerning the nature of mathematics.

Our study shows that most student responses revealed signs of a *process-* and *application*-related orientation, which is a clear indicator that the use of mathematical maps in the geometry course helped to counteract the double discontinuity and promoted favourable beliefs related to the nature of mathematics.

Signs of how such maps may open up a dynamic understanding of mathematics could be seen in, for instance, the students' reasoning about the struggle when mathematicians develop mathematical concepts. The dynamic understanding of mathematics, emphasising the subject as an *emerging science* in which failures belong, was expressed, for example, in terms of how the "crooked development of mathematics" gives students a deeper knowledge of mathematics and its nature.

The implemented parts of the map were perceived as useful and mostly easy to use. Most students showed an application-related orientation in relation to their future role as teachers when using the map, and they used it as a subject didactical tool to motivate their future students to seek the relevance of mathematics.

Winsløw and Grønbæk (2014) classified Klein's double discontinuity into three intertwined dimensions, namely *the institutional context* (of university v school), *how the subject's role differs between institutions* (a student at university or school v a teacher of school mathematics) and *the difference in mathematical content* (elementary v scientific). In this study, we have shown examples of how these three dimensions were present in the students' reflections on the assessment tasks. Hence, the map and the design of the educational context of the geometry course in teacher education for upper secondary teachers (*ämneslärare*) made it possible to foreground aspects of favourable beliefs (Felbrich et al., 2008), and the perceived usefulness of the map revealed students' attitudes and actual system use (Davis, 1985). This is in line with Klein's (1908/1932) focus on the third dimension concerning the differences in content (school mathematics v university mathematics). Klein proposed new approaches to teaching at university level, bridging these gaps. Our findings suggest that using the map may support the development of seeing mutual connections between problems in various fields and make it possible for prospective teachers to connect these to the content in school mathematics.

To overcome Klein's double discontinuity, students need opportunities and support to make connections between the axiomatic-formal world of university mathematics and the perceptual-symbolic or conceptual-embodied world of school mathematics (cf. Tall, 2008). The geometry course (on which this study was based) had been developed based on the formal approach, starting from selected axioms and making logical deductions to prove theorems. The purpose of the project was two-fold. On the one hand, the aim was to help students overcome the journey from the embodied to the formal world (that most students face in university courses) – in other words, to overcome the first discontinuity. This purpose was supported by the use of the map, and many students described in their free-text answers the tool's utility in the process of understanding. We could also follow this process in the PP assignment, for which most of the students had successful results.[3] On the other hand, the introduction of the interactive mathematical map was made with the hope that the students would overcome the gap between the axiomatic-formal environment and the world of school mathematics, where they hopefully will transfer the gained knowledge to their future pupils – in other words, to overcome the second discontinuity. Again, the students' reflections and remarks in their free-text answers sustain the idea that the use of the map provides help in overcoming this gap. However, some students expressed critical remarks that need more attention in further research. In addition, the ND assignment showed that almost 80 % of the students found the content of the map useful for writing a narrative motivation, and 95 % wished to incorporate narrative elements of it in their future teaching; they experienced them as useful in the transfer from the axiomatic-formal world to the perceptual-symbolic and conceptual-embodied world.

Kubli (1999, 2002) reported that students react in a much more positive way to historical material if it is prepared and used in a narrative form. With respect to the teaching process, the methodological use of narrative didactics, which are authentic excerpts from the historical context of the history of mathematics and related course assignments, helped the students to see the vitality of science, which shows itself in research, discovery and creativity (Klassen, 2006). Our findings show that using narrative elements in the teaching process engages students emotionally with the narrative, and therefore also with the mathematical content. This, we argue, is a promising way to design teaching with digital tools, since students in such settings can in a useful way notice connections between the axiomatic-formal world of university mathematics and the perceptual-symbolic or conceptual-embodied world of school mathematics.

Conclusion

Overall, the current study shows that the implemented parts of the map were perceived as useful and mostly easy to use. However, minor shortcomings in the functionality, and especially in the design, became apparent, which will have to be adjusted in a future design cycle. In addition, thanks to the participants' proposals, the map will be enhanced with some helpful functionalities.

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References

Ableitinger, C., Hefendehl-Hebeker, L. & Herrmann, A. (2013). Aufgaben zur Vernetzung von Schul- und Hochschulmathematik. In H. Allmendinger, K. Lengnink, A. Vohns & G. Wickel (Eds.), *Mathematik verständlich unterrichten* (pp. 217–233). Springer Spektrum. doi: 10.1007/978-3-658-00992-2_14

Ausubel, P. D. (1963). *The psychology of meaningful verbal learning*. Grune & Stratton.

Bauer, T. & Partheil, U. (2009). Schnittstellenmodule in der Lehramtsausbildung im Fach Mathematik. *Math Semesterber*, 56, 85–103.

Brandl, M. (2009). The vibrating string – an initial problem for modern mathematics; historical and didactical aspects. In I. Witzke (Ed.), *Mathematical practice and development throughout history: proceedings of the 18th novembertagung on the history, philosophy and didactics of mathematics* (pp. 95–114). Logos.

Brandl, M. (2016). Narrative Mathematik-Didaktik mittels Elementen bildender Kunst. In Institut für Mathematik und Informatik der Pädagogischen Hochschule Heidelberg (Eds.), *Beiträge zum Mathematikunterricht 2016* (pp. 1415–1418). WTM-Verlag. doi: 10.17877/DE290R-17645

Brandl, M. (2017). Narrative Didaktik als Vernetzungsinstrument: die Schule von Athen. In T. Borys, M. Brandl & A. Brinkmann (Eds.), *Mathe vernetzt – Anregungen und Materialien für einen vernetzenden Mathematikunterricht* (pp. 7–20). MUED.

Bruner, J. (1986). *Actual minds, possible worlds*. Harvard University Press. https://archive.org/details/actualmindspossi00jero

Davis, F. D. (1985). *A technology acceptance model for empirically testing new end-user information systems: theory and results* [Ph D thesis]. Massachusetts Institute of Technology]. http://hdl.handle.net/1721.1/15192

Egan, K. (1989a). The shape of the science text: a function of stories. In S. de Castell, A. Luke & C. Luke, (Eds.), *Language, authority and criticism: readings on the school textbook* (pp. 96–108). Falmer Press.

Egan, K. (1989b). Memory, imagination, and learning: connected by the story. *Phi Delta Kappan*, 70 (6), 455–473.

Ernest, P. (2014). What is mathematics, and why learn it? In P. Andrews & T. Rowland (Eds.), *Masterclass in mathematics education: international perspectives on teaching and learning* (pp. 3–14). Bloomsbury.

Felbrich, A., Kaiser, G. & Schmotz, C. (2012). The cultural dimension of beliefs: an investigation of future primary teachers' epistemological beliefs concerning the nature of mathematics in 15 countries. *ZDM*, 44, 355–366. doi: 10.1007/s11858-012-0418-x

Felbrich, A., Müller, C. & Blömeke, S. (2008). Epistemological beliefs concerning the nature of mathematics among teacher educators and teacher education students in mathematics. *ZDM*, 40, 763–776. doi: 10.1007/s11858-008-0153-5

Hefendehl-Hebeker, L. (2013). Doppelte Diskontinuität oder die Chance der Brückenschläge. In C. Ableitinger, J. Kramer & S. Prediger (Eds.), *Zur doppelten Diskontinuität in der Gymnasiallehrerbildung. Konzepte und Studien zur Hochschuldidaktik und Lehrerbildung Mathematik* (pp. 1–15). Springer. doi: 10.1007/978-3-658-01360-8_1

Isaev, V. & Eichler, A. (2017). Measuring beliefs concerning the double discontinuity in secondary teacher education. In T. Dooley & G. Gueudet (Eds.), *Proceedings of CERME 10* (pp. 2916–2923). DCU Institute of Education and ERME. https://hal.archives-ouvertes.fr/hal-01949039

Klassen, S. (2006). A theoretical framework for contextual science teaching. *Interchange*, 37 (1-2), 31–61. doi: 10.1007/s10780-006-8399-8

Klein, F. (1932). *Elementary mathematics from an advanced standpoint*. Macmillan. [Original work published 1908] https://archive.org/details/elementarymathem032765mbp

Klein, F. (2016). *Elementary mathematics from higher standpoint. Volume I: arithmetic algebra analysis*. Springer. [Original work published 1924] doi: 10.1007/978-3-662-49442-4

Koch, M., Confrey, J., Clark-Wilson, A., Jameson, E. & Suurtamm, C. (2021). Digital maps of the connections in school mathematics: three projects to enhance teaching and learning. In A. Clark-Wilson, A. Donevska-Todorova, E. Faggiano, J. Trgalová & H.-G. Weigand (Eds.), *Mathematics education in the digital age: learning practice and theory* (pp. 121–137). Routledge. doi: 10.4324/9781003137580

Kubli, F. (2002). *Plädoyer für Erzählungen im Physikunterricht: Geschichte und Geschichten als Verstehenshilfen – Ergebnisse einer Untersuchung* (2nd ed.). Aulis Deubner.

Kubli, F. (1999). Historical aspects in physics teaching: using Galileo's work in a new Swiss project. *Science & Education*, 8 (2), 137–150. doi: 10.1023/A:1008613706212

Norris, S., Guilbert, M., Smith, M., Shahram, H. & Phillips, L. (2005). A theoretical framework for narrative explanation in science. *Science Education*, 89, 4, 535–554. doi: 10.1002/sce.20063

Pinto, A. & Cooper, J. (2022). The road not taken – investigating affordances of infinitesimal calculus for enriching the repertoire of secondary mathematics teachers. *International Journal of Research in Undergraduate Mathematics Education*, 8 (2), 318–338. doi: 10.1007/s40753-021-00161-w

Przybilla, J., Brandl, M., Vinerean, M. & Liljekvist, Y. (2021). Interactive mathematical maps – a contextualized way of meaningful learning. In G. A. Nortvedt, N. F. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi et al. (Eds.), *Bringing Nordic mathematics education into the future. Proceedings of NORMA 20* (pp. 209–216). SMDF. https://www.uv.uio.no/ils/english/about/events/2021/norma/proceedings

Przybilla, J., Brandl, M., Vinerean, M. & Liljekvist, Y. (2022). Digital mathematical maps – results from iterative research cycles. In J. Hodgen, E. Geraniou, G. Bolondi & F. Ferretti. (Eds.), Proceedings of CERME12 (pp. 1–8). Free University of Bozen-Bolzano and ERME. https://hal.archives-ouvertes.fr/hal-03754749/

Resnick, L. B. & Resnick, D. P. (1992). Assessing the thinking curriculum: new tools for educational reform. In B. R. Gifford & M. C. O'Connor (Eds.), *Changing assessments: alternative views of aptitude, achievement and instruction* (pp. 37–75). Kluwer. doi: 10.1007/978-94-011-2968-8_3

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula, T. Buttery & E. Guyton (Eds.), *Handbook of research on teacher education* (2nd ed., pp. 102–119). Macmillan. doi: 10.12691/education-2-12-14

Scherer, R., Siddiq, F. & Tondeur, J. (2019). The technology acceptance model (TAM): a meta-analytic structural equation modeling approach to explaining teachers' adoption of digital technology in education. *Computers & Education*, 128, 13–35. doi: 10.1016/j.compedu.2018.09.009

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, 20(2), 5–24. doi: 10.1007/BF03217474

Winsløw, C. & Grønbæk, N. (2014). Klein's double discontinuity revisited: contemporary challenges for universities preparing teachers to teach calculus. *Recherches en Didactique des Mathématiques*, 34(1), 59–86. https://www.researchgate.net/publication/243963587

Wood, D., Bruner, J. & Ross, G. (1976). The role of tutoring in problem-solving. *Journal of Child Psychology and Psychiatry*, 17, 89–100. doi: 10.1111/j.1469-7610.1976.tb00381.x

Notes

1 The weekly assignments are freely available at https://docs.google.com/document/d/e/2PACX-1vTq_w6A2MqresPr5PQLGujDt3NIS6i7aiL7ShsmaR841TtI-hrsyTM2zLY6UEhbULBumIx0zEtTOz6V/pub

2 Italic emphasis in the students' answers was added by the authors.

3 For those students who did not have successful results, more research is needed to understand their perspectives.

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