SUMMARY AND CONCLUSIONS

a) Plan of the Investigations

An investigation must be built up systematically, for which reason it may seemingly consist of a number of parts. This treatise thus consists of the following "sections":

I: A part where the problems and methods are analysed. It consists of chapter I, where a definition is given of the mathematical ability and where some problems concerning it are discussed, and chapters II—IV, which are methodological and historical chapters.

II: The author's conception of the mathematical ability must be translated into tests. This part consists of chapter V, where the tests are presented, and parts of chapters VI and VII, where the problems and aims of the studies are discussed.

III: The author made a first factorial analysis, which is reported in chapter VI, part I.

IV: The results must be checked. In the second part of chapter VI the influence from the properties of the sample is studied, and in the second part of chapter VII the factors are directly checked under new conditions.

V: We must go back to the problem. Have we used an instrument which is sufficiently good? Do we measure the mathematical ability by means of our tests or merely a "test ability"? In chapter VIII factor scores are compared with school marks for mathematics in order to find out if they measure a common aspect of the mathematical ability.

Besides studying the structure of the mathematical ability, the author has attacked a number of other problems. In chapter III the nature of factors and their interpretation are discussed. In chapter VII the first part of the chapter is devoted to the numerical factor.

The nature of the other factors is discussed. A short discussion is given to the possibility of using the present tests as a prognostic instrument of the mathematical ability.

b) Results of the Investigations

By means of the present studies the author wanted to study the structure of the mathematical ability, and at the same time to treat a number of other problems. The starting-point when planning the studies was the following fundamental suppositions:

I: The mathematical ability consists of several aspects. Here we primarily deal with problem solving.

II: In chapter I the mathematical ability is defined in the following way: It is the ability to understand the nature of mathematical (and similar) problems, symbols, methods and proofs; to learn them, to retain them in the memory and to reproduce them; to combine them with other problems, symbols, methods and proofs; and to use them when solving mathematical (and similar) tasks. — The mathematical ability is thus the ability to manage the mathematical subject in school or elsewhere. Thus the term is given a unique definition only if we know the stage at which the ability is investigated and know the population the study refers to. We are, according to the definition, not allowed to be contented with having studied the tests alone.

III: The mathematical ability is only one part of the total intellectual field. Therefore it cannot be successfully studied in isolation but only in connexion with other parts of the field.

The author has translated his conception of the ability to a first battery of 36 tests. Some of the tests are mathematical; we have inductive arithmetical problems, ordinary arithmetical problems, geometrical problems, equations, etc. Other tests define neighbouring domains; we have numerical tests, number series and form series tests, analogies, verbal tests, spatial-perceptual tests, and deductive tests, etc. — In the first study the sample of subjects consisted of 217 boys from two grade levels of high school. The boys were as a rule 14 or 15 years old. The first factor analysis (the Alpha Study) gave six centroid factors. After rotation five of them are interpretable, while the sixth must be treated as a residual factor.

One of the factors is called N, the numerical factor. It is highly loaded with the numerical tests, and shows saturations for a test of equations, for a test where the subjects are to find out which answer to an item of division is correct, and for a non-numerical test with the following instruction: Write after each pair of letters, e.g. FG, the latter letter if they are in alphabetical order, the former letter if they are not in alphabetical order. Thus it was shown that the numerical factor is not restricted merely to computations.

The second factor, V, seems to be defined in the same way as the well-known verbal comprehension factor.

The third factor is visual. It was found in all tests which require the manipulation of spatial materials, but significant loadings were not found for the geometrical tests. There is a tendency in modern factor analysis to split this factor in two or more. This analysis gives only one visual factor.

The fourth factor is called D, and it was found for tests of quantitative reasoning ("syllogisms"), number series, and number analogies. The loadings on the mathematical tests are most probably not significant.

The fifth factor is called R, the general mathematical reasoning factor. The factor is defined by the mathematical tests, including Equations and Numerical Judgment, which are also loaded on N. Loadings are also found for a couple of visual tests. The factor is not a pure mathematical factor, even though it is dominated by the mathematical tests.

The sixth factor has a faint inductive tendency.

The second-order general factor has its highest loadings for the mathematical tests, some of the verbal tests, and the number series and form series tests, etc.

A certain question at once rises as a consequence of this research. We have used a sample of subjects which belongs to two grade levels. Is the structure sufficiently alike on these levels to allow a combination of the samples? This question is connected with the old question: What will happen if we make a sampling more homogeneous?

In order to answer this question the subjects were grouped in two more homogeneous samples, which were enlarged by one class each. One of the samples consisted of 123 subjects in the classes 5^5 and

R I[4], the other of 161 subjects in the classes 3[4] and 4[5]. Eighteen tests were chosen and two new factorial studies were performed, Sub-Studies A and B.

The two studies give mainly the same results as the greater Alpha Study. Four of the factors are defined by the same tests as in this analysis. As to factor R, it was refound in Sub-Study A, and it is defined by the same tests as before. In Sub-Study B the cluster of tests, which had in the other studies defined it, is closely knit, but it is localized so close to the hyperplane through the other factors, that the R factor was not refound in this analysis. By means of Lawley's method the author proved that there are really five significant factors in Sub-Study A. It is, however, evident that the factorial structure is very similar in the two analyses, for which reason we seem justified in combining the two samples of subjects. The correlations between the factors are in most cases lower for the sub-studies than for the Alpha Study.

The first investigation answered a number of questions but also gave birth to new ones. We would, for example, like to make a check of the factors, especially factor R. This is more or less necessary since we are not, at the present stage of study, able to calculate how much error factors influence our results. We also want to know more about the psychological nature of the factors. We are especially interested in factors N, S, and D. Factor R is most probably too complex to be fully investigated by a single analysis, and reasoning factors have repeatedly been studied by others. Factor V has also been thoroughly investigated and is at least partly outside the scope of the present investigation. The author thus planned new studies.

The first part of the seventh chapter deals with the numerical factor. A new theory of its nature is presented, tests which might prove or disprove it were constructed. Methods of solving numerical tasks were studied.

Several authors have found that the numerical factor is not restricted to the numerical domain. The psychological nature of the ability behind the factor has, however, not been sufficiently well studied. Coombs tried to prove that the nature of the numerical factor lies in the ability to manipulate rapidly a well-known symbolic system according to well-practised rules. Coombs's attempts of proving this hypothesis did not succeed.

The present author makes a distinction between a materialistic definition or interpretation of a factor and a functional definition or interpretation. The former deals with properties of the materials of the tests, the latter with properties within the solver or his methods of solving the tasks. With regard to the ability behind the numerical factor, a new theory was built up. According to this theory the numerical ability is primarily dependent on some innate capacity. The numerical ability may, from the materialistic point of view, be characterized as the ability to apply with greater and greater efficiency certain rules on a symbolism, which in the beginning call for reasoning. From the functional point of view it may be characterized as the ability to automatize reasoning. It is a well-known fact that simple tasks at first are solved by means of reasoning, but after a certain time they are solved seemingly without consideration, "automatically". This theory, when proved, would explain why the numerical factor is also of importance to non-numerical tasks.

In order to prove or make probable the latter part of the theory the author compared the introspective statements that were made by the sucjects with their abilities. There was a definite connexion between high numerical ability and reported automatization of the solution of numerical tasks. This is, of course, not a definite proof of the theories. They must be investigated by means of experiments.

From the general hypothesis as to the nature of the numerical ability a number of sub-hypotheses were deduced:

1) A test involving processes, which have been reported to be automatizable, must show successively higher correlations with the numerical tests the more the processes are practised.

2) A test which involves a process that is more easily automatized than the process of another must approach the numerical tests more rapidly than the other.

3) A test, which involves a process that is not successively more automatized by means of practice must not show successively higher correlations with the numerical tasks.

4) A test which involves a process, which has first been automatized, but the automatization of which has been broken, must then show low correlations with the numerical tests.

In order to prove the first sub-hypothesis the ABC-tests were constructed. In order to prove the second sub-hypothesis the ABC-, Syllogisms, and Equations tests were constructed, in order to prove the third sub-hypothesis the Alphabet tests, and in order to prove the fourth sub-hypothesis the 123-test were constructed and included in the studies. The tests are reproduced in Appendix C and described in chapter V.

The correlations between the ABC-tests and the numerical tests rose rapidly towards .70; the correlations between the tests of Syllogisms and the numerical tests also rose, but more slowly; the more easy and uncomplicated test of Equations was more highly correlated with the numerical tests than the more difficult test. The correlations between the Alphabet-test and the numerical tests were nearly constant. The correlations for the 123-test with the numerical tests were of the same size as the correlations for the first ABC-test. Thus the results are in line with the author's sub-hypotheses.

In order to prove further the different sub-hypotheses regarding the numerical ability, to clear up the nature of the other factors, and to check the results of the first factorial studies, a new factorial study, the Beta Study, was performed. Here twenty tests from the first analysis were included. The tests ABC, 123, Syllogisms, Alphabet, and Equations were included to study the numerical factor. Three new spatial-perceptual tests were included to study the visual factor, and the tests Inferences I and II, and Mathematical Reasoning to study the deductive factor. In order to make the study as efficient as possible, the subjects were, in this study, given the tests individually, for which reason it has been possible to study their ways of solving the tasks, etc. The number of subjects was 154 (boys). Most of the subjects were collected among the pupils on grade level 4^5, some, however, on grade levels 5^5 and 3^5.

Eight factors were extracted. Only seven, or perhaps only six of them may be interpreted. The numerical factor was highly correlated with the last (highly practised) ABC-test, and the simple test of equations, and to a lesser degree with the last test of Syllogisms. The correlation with the 123 test was not high. The author's theories were confirmed. The verbal factor was refound. Also the visual factor was isolated, and here, too, only one visual factor was found. An attempt to define the ability behind it was made: It was defined as visual plasticity, i.e. as the ability to effect purposeful and deliberate changes in the properties of visualized, imagined but concrete, objects and structures (conversion, division, or other forms of changes). The deductive factor was found to be different from other deductive factors that have been found by previous authors. It seems to involve quantitative, deductive reasoning. — The general mathematical reasoning factor was refound with loadings of the same magnitude as before. Its existence is thus confirmed. It seems to be related to the capacity of grasping a more complex problem. There was also found a factor common to the two ABC-tests and the 123-test, and a factor common to the two tests of inferences and the two number series tests. The general factor was defined in the same way as before, and the R factor and the mathematical tests are highly dependent on it.

The author intended to study the mathematical ability. A number of tests were constructed for this purpose. We may ask whether the tests really measure important aspects of it. In order to make a comparison between school marks for mathematics and results of the studies, the so called factor scores were computed. The author put together a battery of tests for the different factors, and as a rule took three tests for each of them. Each test was given a weight which will give the battery as high a correlation with the factor in question as possible. As there are two main studies (the Alpha and Beta Studies), there are also two sets of weights for the batteries. It was, however, shown that we may as well use one of them for the other, since the correlations between the estimations would be about .99. The correlations between the factors and the composite tests of the factors were about .90, which is much in comparison with previous results. From these estimated factor scores the correlations between the factors and the school marks for mathematics were found.

The highest correlations were found for factors g and R, especially for the latter. In seven classes the correlations between R and school marks were over .60, in four classes it was over .70, and in one class over .80. Only in two classes the correlations were below .60. Also for factor D the correlations were quite high. In four classes it was over .50. The correlations for factor V were lower, and in most classes they were about .30. The correlations between factor S and school marks were low, except in two classes, where they were .68 and .54, respectively. Two of the correlations between factor N and school marks for mathematics were .60 and .54, respectively, but the other correlations were low.

The results have proved that the general mathematical reasoning factor R and the general factor are closely related to the mathematical ability. The factors D, S, and above all N seem to be important in some classes.

The multiple correlation between the tests and the school marks for mathematics is in most classes about .8, i.e. this correlation is as high as or higher than the correlation between the marks on two successive occasions. We may conclude that we have met with an ability that is very highly correlated with the scholastic mathematical ability.