Students’ strategies of expanding fractions to a common denominator – a semiotic perspective

ANDREAS LORANGE AND REINERT A. RINVOLD

The aim of this article is to identify students’ strategies while solving tasks which involve the expansion of fractions to a common denominator. In this case study we follow two groups of 11 year old students and their use of the artefact multilink cubes in the solution process. The analysis of the students’ strategies is based upon a semiotic-cultural framework. Five different types of strategies are reported: trial-and-error, factual, contextual, embodied-symbolic and symbolic. The concept of semiotic contraction is also used in the analysis.

A lot of research has been carried out involving embodied cognition and the multimodal paradigm (Arzarello & Robutti, 2008; Gallese & Lakoff, 2005; Lakoff & Núñez, 2000; Núñez, 2012; Wilson, 2002). Such studies also encompass gestures and the use of various artefacts. Within the semiotic-cultural framework, learning has been formulated in terms of objectification (LaCroix, 2012; Radford, 2008a). This theory is a foundation for our study, and it will be elaborated in the next section. As far as we know, only one paper (Lorange & Rinvold, 2014) has applied the theory of objectification to physical artefacts in the learning of fractions. In that paper we identified the trial-and-error, factual, contextual, embodied-symbolic and symbolic strategy types of expanding two fractions to a common denominator. The aim of this article is to extend the analysis of these five strategies, and embed the concept of semiotic contractions in the analysis.

We follow two groups of 6th grade students who use the physical artefact *multilink cubes* to solve tasks which involve expanding fractions to a common denominator. Our focus is on how students use these cubes and mathematical signs equipped with a cultural meaning to express and communicate their thinking in social interaction.

Andreas Lorange, *NLA University College*
Reinert A. Rinvold, *Hedmark University College*

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Radford (2010b) has described mathematical thinking in the following way: "[...] thinking is considered a sensuous and sign-mediated reflective activity embodied in the corporeality of actions, gestures, and artifacts" (p. XXXVI). A main point here is that mathematical thinking entails the use of resources located outside of the brain, and that such resources play an important role in mathematical activity.

An essential point in the theory of objectification is that learning is closely connected to actions aimed at noticing different aspects of the mathematical object at hand. We use Radford's definition of a mathematical object: "[...] mathematical objects are f*ixed patterns of reflexive human activity incrusted in the everchanging world of social practice mediated by artifacts*." (2008a, p. 222). This definition emphasises that mathematical objects are patterns of activity closely linked with the use of artefacts. The mathematical object we study is "the procedure of expanding two fractions to a common denominator". This procedure is *a fixed pattern of reflexive human activity*, so it fits well with Radford's definition of a mathematical object. The theory of objectification is used as an analysing tool in order to identify different layers of objectification in the students' strategies for expanding two fractions to a common denominator. In relation to students' generalisation of number patterns, Radford (2006a, 2010a, 2010b) has described the factual, contextual and symbolic layer of generality. Radford also refers to these layers as layers of objectification. These layers are generalised and applied to the students' strategies for expanding fractions. The concept of semiotic contraction will also be central in the analysis, and it will be explicated later. Our research questions were:

> Which strategies do the students employ as they expand two fractions to a common denominator?
>
> Which aspects of the expansion process are at the centre of the students' attention in these strategies?
>
> Which changes or simplifications in the semiotic activity of the students can be observed as their strategies evolve?

Radford's theory of objectification

The theory of objectification (Radford, 2002, 2006b, 2008a) aims to account for the way in which students engage with something in order to notice and make sense of it. The process of objectification is closely linked with actions aimed at bringing something to someone's attention or view (Radford, 2006a, p. 6):

> The term objectification has its ancestor in the word *object*, whose origin derives from the Latin verb *obiectare*, meaning "to throw something in the way, to throw before". The suffix – *tification* comes from the verb *facere* meaning "to do" or "to make", so that in its etymology, objectification becomes related to those actions aimed at bringing or throwing something in front of somebody or at making something apparent – e.g. a certain aspect of a concrete object, like its colour, its size or a general mathematical property.

Since mathematical objects are general, they cannot be fully displayed in the physical world. Therefore the students resort to signs and different kinds of artefacts to express their mathematical experience. The perceptual act of noticing unfolds in a process mediated by a multi-semiotic activity where for example different types of artefacts, spoken words and mathematical symbols are central. These semiotic resources used by the students in the objectification process, Radford (2003, p. 41) calls *semiotic means of objectification*:

> They may manipulate objects (such as plastic blocks or chronometers), make drawings, employ gestures, write marks, use linguistic classificatory categories, or make use of analogies, metaphors, metonymies, and so on. [...] These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, I call semiotic means of objectification.

By focusing on the students' phenomenological experience, the theory of objectification emphasises the subjective dimension of knowing, but the theory also takes account of the social and cultural dimensions of knowing. Through the process of objectification "the students grasp the cultural logic with which the objects of knowledge have been endowed and become conversant with the historically constituted forms of action and thinking" (Radford, 2010b, p. XXXVIII).

The object of knowledge is not a monolithic object, but it is an object that is made up of layers of generality. These layers will be "more or less general depending on the characteristics of the cultural meanings of the fixed pattern of activity in question" (Radford, 2006b, p. 14). A circle, for example, can be expressed through the kinaesthetic movement that forms a circle, the words "a set of points with the same distance to a centre" or through a symbolic formula. Such layers of generality emerge as the student becomes conversant with the mathematical object at hand.

Radford's layers of generality

In relation to students' generalisation of number patterns, Radford (2006a, 2010a, 2010b) has described the factual, contextual and symbolic layers of generality. These layers correspond to generalisation through actions, language and mathematical symbols. Koukkoufis and Williams (2006) have applied Radford's layers of generality to students' generalisation of the compensation strategy in connection with *the dice games instruction method* (Linchevski & Williams, 1999). This method is related to integer addition and subtraction through objects on a model, i.e. red and yellow cubes on a double abacus. Here, we will generalise and apply Radford's layers to the students' strategies for expanding fractions. In this section we will give a short description of these layers of generality, and we start with the factual layer (Radford, 2006a). The students he referred to were to generalise a number pattern which was expressed by a visual representation, see figure 1.

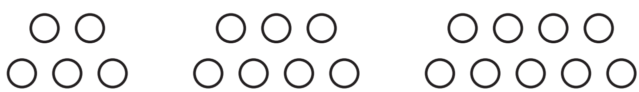

Figure 1. *One of the patterns in Radford's studies*

An example of a way to determine the number of circles in a figure on the factual layer of generality was "one plus one plus three, two plus two plus three, three plus three plus three" (Radford, 2006a, p. 11). Here the circles were grouped in the following way by pointing gestures of the student, see figure 2.

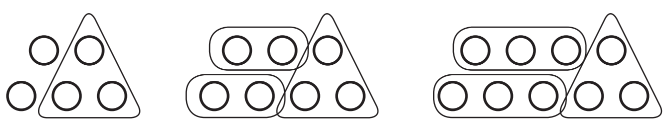

Figure 2. *Grouping of circles on the factual layer*

According to Radford, a factual generalisation is: "a generalization of actions in the form of an operational scheme (in a neo-Piagetian sense). This operational scheme remains bound to the concrete level [...]" (2003, p. 47). In order to refer to actions on physical objects, this operational scheme is generalised through for example deictic semiotic activity like pointing gestures and rhythm.

Like the factual strategy, the contextual strategy also originates from a visual approach to the process of generalisation, but now "the previously constructed operational scheme is generalised through language" (Koukkoufis & Williams, 2006, p. 165). An example of a contextual generalisation of the number pattern shown in figure 1 is: "You double the number of the figure and you add three [...]" (Radford, 2006a, p. 13). A new abstract object, expressed by the formulation "the number of the figure" is introduced and has replaced the previously concrete objects associated with the factual generalisation.

On the symbolic layer of generality, the generalisation is expressed through mathematical symbols, and a formula for the number of circles in a figure is obtained. In connection with the number pattern that is shown in figure 1, an example of a symbolic generalisation is "$n \times 2 + 3$" (Radford, 2006a, p. 14). About this layer of generality Radford (2010a, p. 56) says: "The understanding and proper use of algebraic symbolism entails the attainment of a disembodied cultural way of using signs and signifying through them".

Semiotic contractions

Together with the theory of objectification, the concept of *semiotic contraction* (Radford, 2002, 2008b, 2010a) will be used in the analysis of the students' strategies of expanding fractions to a common denominator. According to Radford, the semiotic contraction is an essential part of the objectification process. As the semiotic contraction takes place, the semiotic activity of the students is condensed into more compact forms. This results in a concentration of meaning and a reduction of the number of actions, words or signs in the mathematical activity which enables the students to focus on the central elements of the mathematical experience (Radford, 2008b, p. 12):

> Contraction [...] makes it possible to cleanse the remnants of the evolving mathematical experience in order to highlight the central elements that constitute it. Contraction is indeed a necessary condition of knowledge attainment. We can easily imagine the difficulties that we would experience if we needed to pay attention to each and every detail of our surroundings and the experience we make of them. We would need to attend to an amazing number of things that go beyond of the threshold of consciousness [...]. Contraction is the mechanism for reducing attention to those aspects that appear to be relevant. This is why, in general, contraction and objectification entail forgetting. We need to forget to be able to focus. This is why to objectify is to see, but to see means at the same time to renounce seeing something else.

In connection with the number pattern in figure 1, a student made the following factual generalisation which we referred to in the previous section: "one plus one plus three, two plus two plus three, three plus three plus three [...]" (Radford, 2006a, p. 11). This utterance was followed by a series of pointing gestures which indicated the grouping of the circles, see figure 2. The grouping of the circles intimated that a pattern of actions underlay this generalisation. Later the student refined this factual generalisation into a contextual generalisation: "You double the number of the figure and you add three [...]" (Radford, 2006a, p. 13). The pattern of action associated with the grouping of circles is no longer in focus, and no gestures were involved. Hence, this is a semiotic contraction. Later the student's generalisation was further contracted into the symbolic generalisation "$n \times 2 + 3$" (Radford, 2006a, p. 14). Here, the symbolic letter "*n*" is the semiotic contraction of the "number of the figure". This formula is "the crystallisation of a semiotic process endowed with its situated history" (Radford, 2006a, p. 14).

We have now described the factual, contextual and symbolic layer of generality and some semiotic contractions that can occur in connection with these layers. Usually, Radford refers to these layers as layers of generality. This is quite natural because these layers emerged in connection with the generalisation of number patterns. Nevertheless, he also refers to these layers as layers of objectification, because his theory of objectification is a general theory. For us, it has been natural to refer to these layers as layers of objectification. The reason for this is that we are not studying the generalisation of number patterns, but our research questions are related to how different aspects of the procedure of expanding two fractions to a common denominator are at the centre of the students' attention in the different strategies they employ. Because our research questions are so tightly connected to Radford's theory of objectification, we will in the following consequently refer to his layers as the factual, contextual and symbolic layer of objectification.

The choice of the theoretical framework

The choice of Radford's theory of objectification as the theoretical framework for our article was made during the analysis of our empirical material. We realised that Radford's descriptions of the factual, contextual and symbolic layers of objectification enable us to describe what we perceive as the most prominent features of the development of strategies we observed in our data. Moreover, Radford's concept of semiotic contractions was suitable to describe the changes and simplifications we observed in the students' strategies as these strategies evolved. We use

Radford's theory in the learning of another mathematical topic than where the theory originally was used, so this may contribute to the generalisation and development of theory. Our choice of framework implies that we emphasise the semiotic aspects of the students' strategies. Radford's theory of objectification focuses to a great extent on the semiotic resources the students use in the learning process. He has defined mathematical objects as "fixed patterns of reflexive human activity incrusted in the everchanging world of social practice mediated by artifacts" (Radford, 2008a, p. 222). We interpret "fixed patterns" as operational schemes which are stable over time and carried out by individuals or a group of individuals in order to solve a problem. The strategies of expanding two fractions to a common denominator were carried out by individuals in social interaction in order to solve the problems they were encountering. The students' strategies stabilised, first on the individual level, and then on the group level. In this sense they became fixed patterns. Radford's definition emphasises that the fixed patterns of activity are mediated by artefacts. Artefacts such as the multilink-cubes and the mathematical signs for fractions play a central role in the students' strategies.

Method

The case study was carried out in a 6th grade classroom in the autumn of 2011 in Norway. All the students came from the same class, and they participated voluntarily. We selected participants according to the framework of *purposive sampling* (Bryman, 2008). This means that we sampled participants in a strategic way, so that those sampled were relevant to our research questions. In cooperation with the teacher we selected two groups of three students who were medium to high achievers. We did not choose low achievers because the students would encounter the expansion of two fractions to a common denominator one year before what is normal in Norwegian schools. Because purposive sampling is a non-probability sampling approach, and the number of participants are so few, this way of sampling does not allow us to generalise our results to a wider population (Bryman, 2008). In spite of this, qualitative analysis based on purposive sampling can deepen our understanding and bring our attention to an issue worth considering (Schoenfeld, 2008).

All the sessions were within the school timetable. Before the sessions started, the students had never encountered the expansion of fractions. Every group had 13 sessions of 45 minutes with one of the researchers. In these sessions the researcher presented a problem which the students were asked to solve. This problem was also given to the students in written form as a task. Normally the students were working on the task without intervention from the researcher. When the students had finished the task, they were asked to explain how they reasoned when they were solving it. Sometimes the students were asked to write down their explanation before they presented it orally. The researcher did not evaluate the utterances of the students. If for example one student in a group had obtained a correct solution and the others had not, the students were asked to discuss between themselves. This worked quite well because the correctness of an answer or the validity of an explanation could be checked against the physical context of the task. For example, one of the introductory tasks was to build a bar which should consist of 30 cubes, of which 4/5 were brown. If the students had built a bar as a solution to this task, it was manageable for them to count the cubes and establish whether 4/5 of the cubes were brown. After the students had presented their solutions and explanations, the researcher could intervene, for example by summarising some important points in the task. Then the students were given a new task to solve, and in this manner the sessions continued. The researcher did not normally teach the students. An exception from this took place in connection with the embodied-symbolic strategy where the researcher showed the students how the expansion of fractions with mathematical symbols could be done and the connection between symbolic expansion and the expansion of bars. This was done because the students had never encountered the expansion of fractions with mathematical symbols.

All of the sessions were videotaped. Research notes and a synopsis were written after each session, and all the salient episodes were transcribed. This constituted the raw material of our analysis which was coded according to the five strategy types we will describe. Radford's framework as described in previous sections was used in the analysis. We chose four excerpts which we will analyse in forthcoming sections. These excerpts were chosen because we consider them to be typical examples of the strategy types we are describing.

The "chocolate bars"

In the beginning the students used multilink cubes to build rectangular "chocolate bars" to depict fractions like 1/2, 1/3 and 2/5. The brown cubes illustrated brown chocolate, and they corresponded to the numerator. The white cubes illustrated white chocolate, and the total number of cubes, regardless of colour, corresponded to the denominator. Figure 3 and 4 show some examples of such bars. The fractions were mainly built as shown in these figures. The students were not explicitly instructed to build the bars this way by the researcher, but our hypothesis is that the students did it this way because it turned out to be a convenient way of building the bars in order to solve the problems they encountered.

We will now define the concepts *strip*, *length*, *height* and *congruent bars*. The word strip was used frequently by the students. The words length, height and congruent bars were not used by the students, but we needed those terms in order to be able to analyse and communicate the students' strategies. These definitions presuppose that the bars are oriented in the same way as in figure 3. A *strip* is a bar with *height* 1. The left part of figure 3 shows a strip which corresponds to 2/5. We say that the *length* of this strip is 5 because it consists of five cubes. If a fraction was to be expanded, the students usually increased the height of the bar. The bar to the right in figure 3 is made up of two strips, and this bar corresponds to the fraction 2/5 expanded by 2. The height equals the expansion factor which is 2. When we use the concept *physical length* or *height*, we do not mean the number of cubes, but the physical measure of the corresponding distance. Two bars are said to be congruent if the corresponding rectangles are congruent, regardless of the colour of the cubes.

Figure 3. *To the left is a strip that corresponds to 2/5. To the right is a bar that corresponds to 2/5 expanded by 2*

In the beginning the students built bars which corresponded to equivalent fractions. An example of this is shown in figure 3. They were also shown different bars and were asked to find the corresponding fractions. Another kind of task was the "completion"-tasks. In these the students were shown a bar which for example consisted of 4 x 2 brown cubes and the task was as follows: "I have begun to build a bar where 1/3 of the cubes are brown. Build the rest of the bar." Another kind of task was: "Build a bar which consists of 30 cubes where 4/5 of the cubes are brown." After these introductory tasks the students started to order two fractions with different denominators by building bars which corresponded to the two fractions. This will be described in the next section.

The trial-and-error strategy type

This strategy was fostered by the researcher, and we will now explain how this was done. The students were shown a rectangular piece of cardboard, and they were asked to solve the following task: "If this was a real chocolate bar, and you could choose between 2/5 or 1/3 of the whole bar, what would you choose?" The students were not able to solve the problem. The instructions to solve the problem which we will now describe were given orally by the researcher and in written form as a task. The students were asked to build some bars where 2/5 of the cubes were brown and put them in a heap and to build some bars where 1/3 of the cubes were brown and put them in another heap (see figure 4 for an example of such bars). Finally they were asked to find two congruent bars, one from each heap and count the brown cubes in the two congruent bars. In this way they found out that 2/5 is greater than 1/3. This strategy corresponds to an elementary layer of objectification because it is a trial-and-error strategy which is more primitive than the strategies that were used later, and in this strategy the students focus on building two congruent bars.

Figure 4. *Bars used to order 2/5 and 1/3 in connection with the trial-and-error strategy*

The factual strategy type

In this section we will describe the factual strategy of expanding two fractions to a common denominator which was frequently used by the students. This strategy was made by the students without any influence from the researcher, and it was more effective than the labour-intensive trial-and-error strategy. We will now give an example of this strategy type. The students were working on the following task:

> Make a chocolate bar where 3/5 of the chocolate is brown, and make another where 2/3 is brown. The bars are to have the same size. Which of the bars has more brown chocolate? Which of the fractions 3/5 and 2/3 is the biggest?

Mary has solved the task by building two congruent bars, and she was asked to pretend that she was a teacher and explain to the others what she had done (pictures are shown in figure 5).

Mary: First you build a strip with three fifths [showing a strip that corresponds to 3/5 , picture 1]. Then you build on as much as you think it shall be. If I for example build four of these [picture 2]. Then you build two thirds [showing a strip that corresponds to 2/3 , picture 3] and see whether it fits or not [picture 4]. So now you have found out how it should fit [removes one of the four strips so that the bar corresponding to 3/5 consist of three strips, picture 5]. Then you enlarge it [expanding the strip that corresponds to 2/3 so that it gets the same size as the bar corresponding to 3/5, picture 6].

Figure 5. *Picture 1 to 3 is in the top row, and picture 4 to 6 in the bottom row*

Mary started with a strip which corresponded to 3/5 . Then she expanded this strip so that the height of the resulting bar became 4. She placed the strip which corresponded to 2/3 upon this bar, and she found out that she had to remove one of the strips to obtain the right height. Finally she placed the strip which corresponded to 2/3 on top of the bar which corresponded to 3/5, and she expanded this strip until the two bars became congruent. Through the semiotic activity the students carried out in connection with the building process, a new aspect of the expansion procedure was thrown in the foreground of the students' attention, namely that the physical heights of the two congruent bars equalled the physical lengths of the strips which corresponded to the fractions that were to be expanded.

We will now argue that the factual strategy type corresponds to a factual layer of objectification. The crucial part of the expansion procedure at this stage is to find the heights of the two congruent bars.

The objectification of this part of the expansion procedure is carried out through actions. By this we mean that artefact-mediated kinaesthetic actions are the semiotic means of objectification in the factual strategy, namely that the physical heights of the two congruent bars are found through the physical lengths of the two strips which correspond to the two fractions. A minor difference is that in Radford's factual layer of objectification the operational scheme which constitutes the factual objectification often consists of pointing gestures and rhythm, but in this case the operational scheme consists of handling of the bars.

The contextual strategy type

In this section we will delineate the contextual strategy of expanding two fractions to a common denominator which was often used by the students. Like the factual strategy, the contextual strategy was not introduced by the researcher, but it emerged as a result of the students' own work. As we will see, the contextual strategy was an improvement of the factual strategy. The following task was given to the students:

> Make a chocolate bar where 2/3 of the chocolate is brown, and make another where 5/7 is brown. The bars are to have the same size. Which of the bars has more brown chocolate? Which of the fractions 2/3 and 5/7 is the biggest?

After the students had built the two congruent bars, they were asked to write down an explanation as to how they built the bars. In the following excerpt, Cathie reads her explanation aloud (see figure 6).

Cathie: First you make a strip with five sevenths [showing a strip that corresponds to 5/7, picture 1]. Then you make another one that shall be two thirds [showing a strip that corresponds to 2/3 , picture 2]. Then you see that on two thirds, that the bottom number is three. So you build three strips with five sevenths [pointing gesture with the pencil, picture 3]. Then you see that the bottom number in five sevenths is seven. Then you take seven lengthwise [gliding pointing gesture with the pencil, picture 4].

Figure 6. *Picture 1 to 4 is ordered from the left to the right*

Cathie started with a strip which corresponded to 5/7, and she expanded this strip so that the height of the resulting bar became 3 because the denominator of the other fraction was 3. Then the strip which corresponded to 2/3 was expanded so that the height of the resulting bar became 7 because the denominator of the other fraction was 7. In the factual strategy the heights of the two congruent bars were found through the physical lengths of the two strips which corresponded to the fractions that were to be expanded. In the contextual strategy these heights were found through the denominators of the fractions which the students referred to as "the bottom numbers". We argue that the semiotic activity carried out in connection with the building process have thrown a new aspect of the expansion procedure into the foreground of the students' attention, namely that the denominators of the two fractions equalled the height of the two congruent bars.

In connection with the factual strategy, the strip which corresponded to the first fraction was expanded by a number of strips according to what the students thought was necessary in order to obtain two congruent bars. Then the physical height of the resulting bar was compared with the physical length of the strip which corresponded to the other fraction. If the physical height of the bar was too long or too short, some strips were removed or added. In the contextual strategy this intermediate stage is no longer carried out, but the two congruent bars are built directly. The operational scheme of the factual strategy is now objectified through language, i.e. the words "the bottom numbers". Therefore we argue that this strategy type corresponds to a contextual layer of objectification. Furthermore, the semiotic activity of the students has been condensed into a more compact form. Consequently, a semiotic contraction has taken place.

The embodied-symbolic strategy type

At this stage of the objectification process the students had never encountered the expansion of fractions with mathematical symbols such as number symbols, multiplication signs and long fraction lines as shown in figure 7. Therefore, the researcher showed them how this could be done and the connection between symbolic expansion and the expansion of bars. After this the students started to add fractions with different denominators, and the embodied-symbolic strategy arose. We will now give an example of this strategy type. The students were working on the following task:

> One Saturday Peter makes a pizza for himself and his friends. When they have eaten, there is 1/3 pizza left which he puts in the freezer.

> The next Saturday he also makes a pizza for his friends. Then there is 2/5 pizza left which he puts in the freezer. How much pizza has Peter frozen after the two Saturdays? [The task was accompanied by a picture of two rectangular pizzas of equal size]

Cathie has solved the task by building two congruent bars, and she has also carried out the calculation by writing down mathematical symbols, see figure 7. Then she was asked to explain what she had done.

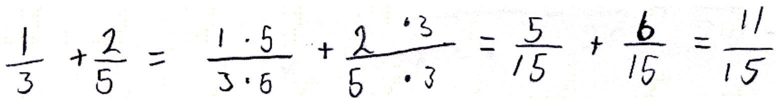

Figure 7. *Cathie's calculation of 1/3 + 2/5.*

Cathie: First I wrote one third plus two fifths. Then the equal sign. Then I wrote one third again with a long fraction line. Then I counted how many I had to expand it to, which was five [gliding pointing gesture along the bar that corresponds to 1/3, picture 1]. Then I wrote times five over and under the fraction line. Then plus two fifths. Then I counted how many strips I had expanded it to [gliding pointing gesture along the bar that corresponds to 2/5, picture 2], which was three. Then I multiplied that [pointing at 1x5/3x5 on her sheet] which became five fifteenths. Plus that [pointing at 2x3/5x3 on her sheet] which was six fifteenths. Which equals eleven fifteenths.

Figure 8. *Picture 1 and 2 is ordered from the left to the right*

In the factual and contextual strategy the students' attention was directed at how to build the two congruent bars, but now the building procedure is no longer in focus, and the students gave no explanation as to how they built the congruent bars. Instead their attention was directed at the connection between the bars and the mathematical symbols. The expansion factors in the symbolical representation of the procedure were found through the heights of the two congruent bars. In this connection the pointing gestures helped the students to focus their attention on the heights of these bars. What is being objectified now is the connection between the bars and the mathematical symbols. Radford (2006a, p. 11) states that the process of objectification is "a matter of endowing the conceptual objects that the student finds in his/her culture with meaning". In this excerpt we see that the mathematical symbols are endowed with meaning through the handling of the multilink-cubes in a process of active elaboration. Furthermore, the written calculation constituted a contraction of the semiotic activity of the students. The lengthy building process is now contracted and condensed into the calculation in figure 7.

The symbolic strategy type

After some time the students found out that it was not necessary to build bars in order to add fractions with different denominators. During one of the sessions Peter suddenly exclaimed: "I think I really understand what I am going to do with this now." Some minutes later when the students were adding 1/4 and 1/5 he again spontaneously exclaimed: "May I say something first. Actually, you don't have to build in a way. No matter how you expand, you don't have to build." At this point Cathie interrupted and said: "In any case you multiply it, one fourth, you multiply it by five. So you multiply it by the opposite denominator." We will now analyse an example of the symbolic strategy type. The following task was given to the students:

> Each of the two brothers Bill and Benny won a chocolate bar at the charity bazaar. Bill gave away 2/3 of his chocolate to his mother, and Benny gave away 1/5 of his chocolate to his mother. How much chocolate did their mother get? [The task was accompanied by a picture of two rectangular chocolates of equal size]

Peter had on his own initiative solved the task without using the multilink-cubes, and he had written down the mathematical symbols shown in figure 9. He had also written down an explanation of what he had done and was asked to read it aloud.

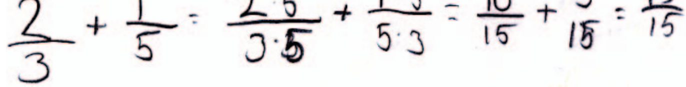

Figure 9. *Peter's calculation of 2/3 + 1/5*

Peter: First I wrote two thirds plus one fifth which equals two thirds with a long fraction line. Then I put times five on top and bottom because the denominator of the other fraction was five. Then I wrote plus one fifth with a long fraction line, times three because the denominator of the other fraction was three.

Now the students' attention was removed from the interplay between the bars and the mathematical symbols, and they focused only on the symbolic representation of the calculation. The symbolic strategy type reported in this section corresponds to a symbolic layer of objectification because the expansion procedure is carried out through mathematical symbols which are used in a disembodied way. The students are no longer referring to the bars.

Conclusion and further research

Different aspects of the procedure of expanding two fractions to a common denominator have been at the centre of the students' attention in the strategy types we have described. In the trial-and-error strategy, the students' attention was directed at building two congruent bars which corresponded to the two fractions that were to be expanded. In the factual strategy, the students objectified the fact that the physical lengths of the strips which corresponded to the two fractions equal the heights of the two congruent bars. This objectification caused the building process to become significantly shortened. In the contextual strategy, the objectification of the relation between the denominators and the heights of the congruent bars resulted in a semiotic contraction because the operational scheme of the factual strategy was now objectified through the words "the bottom numbers". In the embodied-symbolic strategy, the lengthy building process was now contracted into the symbolic representation of the calculation. In connection with the symbolic strategy, the handling of the multilink-cubes and the pointing gestures were no longer part of the semiotic activity of the students. The five strategy types reported here correspond to different layers of objectification, and on these layers the students relate to the mathematical object – "the procedure of expanding two fractions to a common denominator" – in more sophisticated ways.

A question that arises after analysing our empirical material is to what degree Radford's layers of objectification might be used as an analysing tool in connection with other artefacts and other mathematical topics. Because our empirical material gives an example of how Radford's layers can be used as an analysing tool in connection with the expansion of two fractions to a common denominator, our data confirms that Radford's layers can be generalised to other fields. The fact that

Koukkoufis and Williams (2006) describe how Radford's layers emerge in connection with integer addition and subtraction, supports this hypothesis. Still, there is a need for more research to elucidate to what degree these layers of objectification might be generalised to other fields. We have also described the trial-and-error and the embodied-symbolic strategy types. A trial-and-error strategy type is not a new phenomenon, but according to our knowledge, the embodied-symbolic strategy type is not yet described in the research literature. An interesting research question is whether this strategy type can be generalised to other fields, but more research is needed to throw light upon this issue.

The design of learning activities has not been a theme in this article. Nevertheless, it would have been interesting to know more about the transitions between different layers of objectification [by "transition" we mean the process of active elaboration which results in proceeding from one layer of objectification to another]. The transitions to the factual, contextual and symbolic layers of objectification were not initiated by the researcher. Because the building of the congruent bars entailed a lot of work, and the expansion procedure was repeated many times by the students, we conjecture that these transitions arose in order to carry out the expansion procedure in more efficient ways. Still, the transitions between these layers of objectification remain somewhat opaque. We need to know more about how to design learning activities which may facilitate such transitions. Koukkoufis and Williams (2006) suggest that Radford's layers of objectification might raise design related issues. We concur in this view and suggest that Radford's descriptions of the factual, contextual and symbolic layer of objectification can elucidate what kind of actions and learning activities which are suitable to throw new aspects of the mathematical object into the centre of the learner's attention. How the students' mathematical activity thus can be grounded in actions, formulated through language and expressed in mathematical symbols will be investigated further in future research projects.

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Andreas Lorange

Andreas Lorange is Assistant Professor at NLA University College in Bergen. His main research interest is related to how physical and visual artefacts can be used in connection with the learning of mathematics.

andreas.lorange@nla.no

Reinert A. Rinvold

Reinert A. Rinvold is Associate Professor at Hedmark University College in Hamar. His main research interest is multimodal mathematical thinking and learning.

reinert.rinvold@hihm.no