Teaching mathematical problem solving

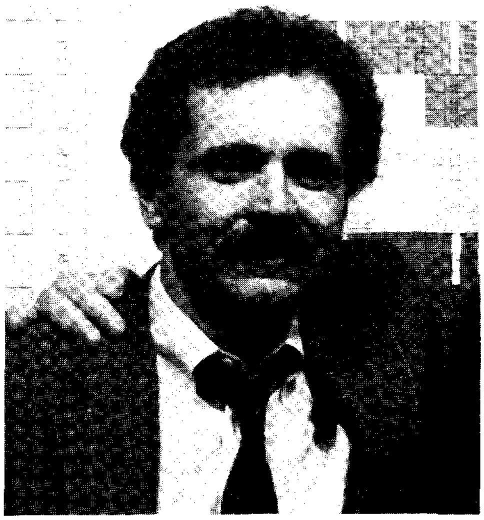
*I november 1987 besöktes lärarhögskolan i Mölndal av en världsberömd problemlösningsexpert, professor **Frank Lester**, Indiana, USA. Hans mycket uppskattade föreläsning publiceras nu i Nämnaren.*

All mathematics educators agree that problem solving is a very important, if not the most important goal, of mathematics instruction at every level. Indeed, some have even gone so far as to insist that *Problem solving should be the focus of school mathematics* (National Council of Teachers of Mathematics, 1980, p. 1).[1] Unfortunately, when this pronouncement was made it was not accompanied by any suggestions as to *how* to make problem solving the focus of instruction. Since about 1980 problem solving has been the most written about, but possibly the least understood, topic in the mathematics curriculum in the United States. It probably is safe to say that most teachers agree that the development of students' problem-solving abilities is a primary objective of instruction. It is equally as evident that these same teachers would admit that it is quite another matter to decide how this goal is to be reached (i.e., where to begin, what problems and problem-solving experiences to use, when to give problem solving particular attention, etc.). Although acceptance of the notion that problem solving should play a prominent role in the curriculum has been widespread, there has been anything but widespread acceptance of *how* to make it an integral part of the curriculum. Indeed, it is common to hear teachers voice concerns like: *As if there wasn't already enough content to cover. Now the "experts" want us to add problem solving.*

Comments of this sort cannot be brushed aside lightly. They point to the fact that to date no mathematics program has been developed that adequately addresses the issue of making problem solving the central focus of the curriculum. Instead of programs with coherence and direction, what teachers have been given is a well-intentioned *mélange* of story problems, lists of strategies to be taught, and suggestions for classroom activities. If problem solving is to become a more prominent goal of mathematics instruction, more serious and thoughtful attention must be given to what it means to make problem solving the focus of school mathematics. Before presenting my ideas for teaching problem solving let me illustrate the severity of the problem facing us as teachers by means of a few examples from my own experience.

[1] The National Council of Teachers of Mathematics (NCTM) is an American professional organization for persons interested in the teaching and learning of mathematics. Currently, the NCTM has approximately 70,000 members.

Why Is Mathematical Problem Solving Difficult for Students?



I have been a mathematics teacher for more than 22 years. During my career I have taught students ranging in age from 6 years to adult (my oldest student was over 60 years old when I taught her). As one would expect, some of these students have been exceptionally talented in mathematics and a few have found mathematics a particularly troublesome subject to learn. But, on the whole, most of my students have been of average ability in mathematics. The examples that follow are taken from my experiences with this large majority of average ability students. As is true of any reasonably serious teacher, I have been puzzled from time to time by the behavior of some of these students. My puzzlement has been nowhere more pronounced than in my observations of students' problem-solving behavior. Consider the following three "problems" and the behaviors of several students who have attempted to solve them.

The Frog in the Well

A frog is at the bottom of a well that is 10 meters deep. During the daytime the frog climbs up the side of the well 4 meters, but at night it slides back 2 meters when it sleeps. At this rate, how many days will it take the frog to reach the top of the well?

It has been my experience that most students (and people in general) over the age of 8 or so determine that it will take the frog *5 days* to reach the top of the well. (Their reasoning goes something like this: 4 m − 2 m = 2 m gain each day and 10 days − 2 m per day = 5 days.) By contrast, children 8 years old or less draw some sort of picture and arrive at either *3 1/2* or *4 days* as their answer (each of which can be considered correct). What happens to students that makes them less successful on this problem as they become older?

The Chickens and Pigs

Tom and Susan went to their grandparents' farm and saw some chickens and pigs in the barnyard. Tom said he saw 18 animals in the barnyard. Susan agreed with him and added that she counted 52 legs in all. How many chickens and how many pigs were in the barnyard?

About six years ago I gave the *Chickens and Pigs* problem to a class of grade three students (ages 8 and 9). Nearly half of them gave *70* as their answer—not *70 animals* or *70 legs,* just *70.* Some of these children pointed out that the words "in all" in the problem tell you to add. A few weeks later I asked a class of grade five students to solve this same problem. Many of them wrote nothing on their papers. When questioned as to why they did not give an answer, typical responses included: *I don't know how to do it,* and *I think you have to divide, but 18 doesn't go into 52.* Not one student in a class of 30 drew any sort of diagram, made any guesses, or otherwise used any "natural" problem-solving strategies. Many of the younger students had been taught a faulty strategy (viz., look for "key words"), but even worse, the older students had actually "learned" that they could not solve mathematics problems.

The Car Trip

> *A man drove his car from his home to a friend's house at a speed of 64 kph and it took him 20 minutes. When he returned to his home he travelled along the same roads but at a speed of 80 kph. How long did the return trip take?*

I have not been particularly puzzled by the fact that many high school and university students fail to solve the *Car Trip* problem successfully. What has puzzled me is that so many (nearly 25%) give *25 minutes* as the answer ($64/20 = 80/x$). Doesn't it defy logic and common sense that if you go faster, it would take longer to make a trip?

Examples such as these are all too common. In fact, most mathematics teachers are quick to observe that many of their students are unable to solve any but the most routine problems despite the fact that their students seem to have "mastered" all of the requisite computational skills, facts, and algorithmic procedures. The first reason why students are often unable to solve any but the most routine problems is that solving a mathematics problem requires the individual to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine. Furthermore, these cognitive actions are influenced by a number of noncognitive factors. That is to say, by its very nature problem solving is an extremely complex form of human endeavour that involves much more than the simple recall of facts or the application of well-learned procedures.

Successful problem solving involves the process of coordinating previous experiences, knowledge, and intuition in an effort to determine an outcome of a situation for which a procedure for determining the outcome is not known.

A second reason why so many students have trouble becoming proficient problem solvers is that they are given too few opportunities to engage in real problem solving. That is, they do not develop expertise in problem solving because they are neither guided nor challenged to do so. Since problem solving is so complex students need to be given carefully designed problem-solving instruction and they must have extensive experiences in solving a wide variety of problems and reflecting on their performance.

The Complex Nature of Mathematical Problem Solving

The ability to solve mathematics problems develops slowly over a very long period of time because it requires much more than merely the direct application of some mathematical content knowledge. Problem-solving performance seems to be a function of at least five broad, interdependent categories of factors:

(1) *knowledge acquisition and utilization*
(2) *control*
(3) *beliefs*
(4) *affects*
(5) *socio-cultural contexts.*

Let me say a few words about each of these categories.

Knowledge Acquisition and Utilization

It is safe to say that the overwhelming majority of research in mathematics education has been devoted to the study of how mathematical knowledge is acquired and utilized. By "knowledge" I mean both informal and intuitive knowledge as well as formal knowledge. Included in this category are a wide range of resources that can assist the individual's mathematical performance. Especially important types of resources are the following: facts and definitions (e.g., 7 is a prime number, a square is a rectangle having 4 congruent sides), algorithms (e.g., the long division algorithm), heuristics (e.g., drawing pictures, looking for patterns, working backwards), problem schemas (i.e., packages of information about problem types), and the host of routine, but not algorithmic, procedures that an individual can bring to bear on a mathematical task (e.g., procedures for solving equations, general techniques of integration). Of particular significance to this discourse is the way individuals organize, represent, and ultimately utilize their knowledge. There is not doubt but that many problem-solving deficiencies can be attributed to the existence of "unstable conceptual systems" (Lesh, 1985). That is, when individuals are engaged in solving a problem it is likely that at least some of the relevant mathematical concepts are at intermediate stages of development. In such cases problem solvers must adapt their concepts to fit the problem situation. To the extent that they are able to make appropriate adaptations, they are successful in solving the problem.

Control

Control refers to the marshalling and subsequent allocation of available resources to deal successfully with mathematical situations. More specifically, it includes executive decisions about planning, evaluating, monitoring, and regulating. Two aspects of control processes have become increasingly popular as objects of research in recent years: *knowledge about* and *regulation of* cognition. The processes used to regulate one's behavior are often referred to as metacognitive processes, and these have recently become the focus of much attention within the mathematics education research community. In fact, recent research suggests that an important difference between successful and unsuccessful problem solvers is that successful problem solvers are much better at controlling (i.e., monitoring and regulating) their activities. It is clear that a lack of control can have disastrous effects on problem-solving performance.

Beliefs

Schoenfeld (1985) refers to beliefs, or "belief systems" to use his term, as the individual's mathematical world view; that is, ". . . the perspective with which one approaches mathematics and mathematical tasks" (p. 45). Beliefs constitute the individual's subjective knowledge about self, mathematics, the environment, and the topics dealt with in particular mathematical tasks. For example, my colleagues and I have found that many elementary school children believe that *all* mathematics story problems can be solved by direct application of one or more arithmetic operations, and which operation to use is determined by the "key words" in the problem (Lester & Garofalo, 1982). It seems apparent that beliefs shape attitudes and emotions and direct the decisions made during mathematical activity. In my own research I have been particularly interested in students' beliefs about the nature of problem solving as well as about their own capabilities and limitations (Lester, Garofalo & Kroll, in press).

Affects

This domain includes individual feelings, attitudes and emotions. Mathematics education research in this area often has been limited to examinations of the correlation between attitudes and performance in mathematics. Not surprisingly, attitudes that have been shown to be related to performance include: motivation, interest, confidence, perseverance, willingness to take risks, tolerance of ambiguity, and resistance to premature closure.

To distinguish between attitudes and emotions I choose to regard attitudes as traits, albeit perhaps transient ones, of the individual, whereas emotions are situation-specific states. An individual may have developed a particular attitude toward some aspect of mathematics which affects her or his performance (e.g., a student may greatly dislike problems involving percents). At the same time, a particular mathematics task may give rise to an unanticipated emotion (e.g., frustration may set in when a student finds that he or she has made little progress toward solving a problem after working diligently on it for a considerable amount of time). The point is that an individual's performance on a mathematics task is very much influenced by a host of affective factors, at times to the point of dominating the individual's thinking and actions.

Socio-Cultural Contexts

In recent years, the point has been raised within the cognitive psychology community that human intellectual behaviour must be studied in the context in which it takes place (Neisser, 1976; Norman, 1981). That is to say, since human beings are immersed in a reality that both affects and is affect-ed by human behaviour, it is essential to consider the ways in which socio-cultural factors influence cognition. In particular, the development, understanding, and use of mathematical ideas and techniques grow out of social and cultural situations. D'Ambrosio (1985) argues that children bring to school their own mathematics which has developed within their own socio-cultural environment. This mathematics, which he calls "ethnomathematics," provides the individual with a wealth of intuitions and informal procedures for dealing with mathematical phenomena. Furthermore, one need not look outside the school for evidence of social and cultural conditions that influence mathematical behavior. The interactions that students have among themselves and with their teachers, as well as the values and expectations that are nurtured in school, shape not only what mathematics is learned, but also how it is learned and how it is perceived (cf., Cobb, 1986). The point then is that the wealth of socio-cultural conditions that make up an individual's reality plays a prominent role in determining the individual's potential for success in doing mathematics both in and out of school.

These five categories overlap (e.g., it is not possible to completely separate affects, beliefs, and socio-cultural contexts) and they interact in a variety of ways too numerous to name in these few pages (e.g., beliefs influence affects, and they both influence knowledge utilization and control; socio-cultural contexts have an impact on all the other categories). It is perhaps due to the interdependence of these categories that problem solving is so difficult for students.

Teaching Problem Solving

For students who are struggling to become better problem solvers the difficulty caused by the complexity of problem solving is compounded by the fact that most of them do not receive adequate instruction, either in quality or quantity. Since problem solving is so complex, it is difficult to teach. Unfortunately, we do not yet have fool-proof, easily followed and implemented methods of helping students to improve their problem-solving ability.

In recent years there has been much research conducted on various approaches to mathematical problem-solving instruction. I will not discuss this research here except to say that detailed discussions of the research conducted in the United States during the past 15 years can be found in Lester (1980, 1983), Schoenfeld (1985) and Silver (1985). In the next section of this paper I present my analysis of what this research *suggests* to me about how mathematical problem solving should be taught.

Fundamental Principles About Teaching Problem Solving

In my study of the research literature I was able to isolate four basic principles that stood out as common results of all of the research. These principles are as follows.

I. Students must solve many problem in order to improve their problem-solving ability.
II. Problem-solving ability develops slowly over a prolonged period of time.
III. Students must believe that their teacher thinks problem solving is important in order for them to benefit from instruction.
IV. Most students benefit greatly from *systematically* planned problem-solving instruction.

I will not elaborate on the first three of these principles except to mention that although many factors are necessary ingredients for a successful problem-solving program, perhaps none is more important than principle III. Teachers must demonstrate enthusiasm for problem solving and communicate through their actions and words the importance of problem solving in mathematics. When teachers make a sincere commitment to developing students' problem-solving skills, students will make a similar commitment. In the remainder of this paper I will discuss what is involved in "systematically planned problem-solving instruction."

Essential Components of a Systematically Planned Problem-solving Program

Most problem-solving programs will seem to work for a while in the classroom. However, for a program to be successful all year and year after year, it should be made up of three components: (a) appropriate content, (b) a teaching strategy, and (c) guidelines for managing the program. Let us look at each component in turn.

A. Appropriate Content

First and foremost, a good problem-solving program must include appropriate content. The content must be of suitable difficulty and must include at least three types of experiences designed to improve problem-solving performance. These types of experiences are the following: (1) regular sessions devoted to solving a variety of kinds of problems; (2) instruction in the use of various problem-solving strategies; and (3) practice aimed at the development of specific problem-solving thinking processes and skills.

The focus of instruction should be on the solution of "process" problems, but routine one-step verbal problems and multiple-step verbal problems should also be included. Briefly, a process problem is one whose solution cannot be obtained simply by performing computations. Such problems are included because they exemplify the processes inherent in thinking through and solving a *true* problem (i.e., a situation for which a procedure for solving the problem is not readily at hand). These types of problems serve to develop general strategies for understanding, planning, and solving problems, as well as evaluating attempts at solutions.

One reason why students have difficulty with problem solving is that many of them have not been taught how to use specific problem-solving strategies. Traditionally, most students are taught only one strategy—choose an operation or operations to perform, then do the computations. Several of the strategies that should be included in instruction in grades 1-8 are the following:

- choose an operation or operations to perform
- draw a picture
- make an organized list
- write an equation
- act out the situation
- make a table or chart
- guess and check
- work backwards
- solve a simpler problem
- use objects or models

Instruction aimed at developing students' ability to use strategies such as these needs to include two phases. During the first phase students are taught *how* to use a particular strategy. This phase emphasizes the meaning of a strategy and the techniques involved in implementing it. After students are introduced to a strategy they are given practice using it to solve problems. The second phase is where students are taught to decide *when* to use a strategy. Here students are given problems to solve but they are not told which strategy to use. They must select from among the strategies they have learned the one(s) that are appropriate for solving a given problem. Both phases must be included in instruction. Unfortunately, it is beyond the scope of this paper to elucidate effective ways to coordinate these two phases.

A person who is learning to play a musical instrument, say the piano, does not learn to play simply by playing musical scores. In addition, considerable time must be devoted to activities designed to help her or him master certain skills and techniques. Such skill activities (e.g., finger exercises) are an essential part of becoming an accomplished pianist. In a similar manner, the novice problem solver must be asked to do more than simply solve problems. The third of the three types of experiences involves activities designed to develop certain problem-solving thinking processes and skills. In my own work I have identified 8 thinking processes that are involved in the solution of mathematics problems. A good problem-solving program includes numerous activities that focus on these thinking processes and the skills associated with them. A list of the thinking processes is given in Table 1.

Table 1: Mathematical Problem-solving Thinking Processes
(taken from Charles, Lester & O'Daffer, 1987)

1. Understanding/formulating the question in the problem/situation.
2. Understanding the conditions and variables in the problem.
3. Selecting/finding data needed to solve the problem.
4. Formulating subproblems and selecting appropriate solution strategies to pursue.
5. Correctly implementing the solution strategy and attaining subgoals.
6. Giving an answer in terms of the data given in the problem.
7. Evaluating the reasonableness of the answer.
8. Making appropriate generalizations.

Finally, in addition to the three types of experiences discussed above, this content should be integrated throughout the entire mathematics program. (Occasional attention to problem solving or including a short unit of instruction on problem solving simply is not enough).

B. A Specific Teaching Strategy

Teachers must have a specific strategy for teaching the content. A teacher who is not aware of specific ways to teach problem solving often resorts to general admonitions for students to do better when they need assistance. Comments like: "Read the problem again," "Use your head," and "Think harder" are commonly made by such a teacher. Teacher comments such as these may encourage students to try harder, but they are of little help to students who are truly in need of help. Indeed, very few students can become successful problem solvers without the aid of their teachers. The single most challenging task for the teacher is to decide what kind of guidance to provide and when to provide it. The teacher must play an active part during classroom problem-solving activities by observing, questioning, and, if necessary, by providing direction. During the past fifteen years I have collaborated with several colleagues in the development of a teaching strategy for problem solving. This strategy has been tested and shown to be successful in quite a large number of classrooms in a variety of schools throughout the United States (see Lester, 1983 and Charles & Lester, 1984 for discussions of the results of our research). These "teaching actions" as we call them can be used in a teacher-directed activity, beginning with a whole-class discussion of the problem, followed by individual or small-group work on the problem, and ending with another whole-class discussion of the problem. A list of the teaching actions and the purpose of each action are shown in the following table. These teaching actions cannot be used as a "formula" that will guarantee success for all students. Rather, they provide teachers with a means to guide students systematically through the process of solving problems in order to build confidence as well as competence. As students become more capable as problem solvers the teacher's overt role diminishes.

Table 2: Teaching Actions for Problem Solving

(taken from Charles & Lester, 1982)

	Teaching Action	Purpose
	Class Discussion BEFORE	
1	Read the problem to the class or have a student read it. Discuss vocabulary and the setting of the story as needed.	Illustrate the importance of reading problems carefully for good understanding.

2	Ask questions related to understanding the problem. Focus on what the problem is asking and which data are needed to solve the problem.	Focus attention on important data in the problem and clarify confusing parts of the problem.
3	Have students suggest possible solution strategies. Do not censor or evaluate ideas at this time. (As students become more successful, this action may be eliminated.)	Elicit ideas for possible ways to solve problems. Encourage flexible thinking and exploration.

DURING Students' Solution Efforts

4	Observe students as they solve the problem. Ask them questions about their work.	Diagnose students' strengths and weaknesses in problem solving. Develop reflectiveness about their work.
5	Provide hints for students who are hopelessly stuck or who are becoming too frustrated. Repeat understanding questions as needed.	Help students past insurmountable obstacles in solving a problem and reduce the chances of the development of negative attitudes. Help them learn *how* to use particular strategies.
6	When students get an answer, require them to check their work against data in the problem.	Develop students' ability to evaluate their work.
7	Give students who finish early a variation of the original (do this with all students as time permits).	Help students learn to generalize their solutions.

Class Discussion AFTER

8	Discuss students' solutions to the problem. Identify different ways the problem might be solved.	Help students learn *when* to use a particular strategy and *how* to use it efficiently. Encourage flexibility in solving problems.
9	Compare the problem just solved with problems solved previously and discuss any variations that may have been solved.	Foster transfer of learning.
10	Discuss any special features of the problem such as misleading information.	Help students recognize problem features that influence how problems are solved.

C. Guidelines for Managing the Program

In addition to knowing what and how to teach, the teacher should be provided specific suggestions regarding the management of the program. In particular, the teacher must know how to deal with issues such as:

1) the amount of time to devote to problem solving
2) ways to group students for instruction (allowing students to work on problems in groups of three or four has been shown to be quite successful)
3) adjusting instruction for high and low achievers in the same classroom
4) how to evaluate students' performance
5) how to create and maintain a positive classroom climate for problem solving. There is so much involved in this final component that it could easily be the topic of a separate paper. In fact, recently I collaborated with two colleagues in the preparation of a book on the single topic of evaluation of students' problem-solving performance (Charles, Lester & O'Daffer, 1987). More information about this component can be found in the evaluation book or in a book written earlier by R. Charles and me (Charles & Lester, 1982).

Closing Comments

In this short paper I have attempted to provide some perspective about the nature of mathematical problem solving and I have offered several suggestions as to how to begin to implement a mathematics program that has problem solving at its core. Yes, mathematical problem solving is difficult for children to do and it is difficult for teachers to teach. However, helping children to be better problem solvers in mathematics is not only an extremely important goal, it is also the most challenging and exciting one that a teacher can have. If I were allowed to give only one bit of advice to a teacher who was planning to begin to make problem solving the focus of instruction, it would be to remember that children are natural problem solvers. The teacher's job is to try to develop this natural ability to its maximum extent and to add to the already extensive repertoire of problem-solving techniques that children have at their disposal.

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