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ICT in mathematics education: the future and the realities

Proceedings of MADIF 10
The tenth research seminar of the
Swedish Society for Research in
Mathematics Education
Karlstad, January 26–27, 2016

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Preface

This volume contains the proceedings of *MADIF 10*, the tenth Swedish Mathematics Education Research Seminar, held in Karlstad, January 26–27, 2016. The theme for this seminar was *ICT in mathematics education: the future and the realities*. The MADIF seminars are organised by the Swedish Society for Research in Mathematics Education (SMDF). MADIF aims to enhance the opportunities for discussion of research and exchange of perspectives, amongst junior researchers and between junior and senior researchers in the field. The first seminar took place in January 1999 at Lärarhögskolan in Stockholm and included the constitution of the SMDF. The second meeting was held in Göteborg in January 2000, the third in Norrköping in January 2002, the fourth and fifth in Malmö in January 2004 and 2006, respectively, and the sixth and seventh in Stockholm in January 2008 and 2010, respectively. MADIF 8 and MADIF 9 were held in Umeå. Printed proceedings of the seminars are available for all but the very first meeting. This volume and the proceedings from MADIF 9 are also available in digital form and work is being done to make previous version digitally available.

The members of the MADIF 10 programme committee were Johan Häggström (University of Gothenburg, chair), Eva Norén (Stockholm University), Jorryt van Bommel (Karlstad University), Judy Sayers (Stockholm University), Ola Helenius, (NCM, University of Gothenburg) and Yvonne Liljekvist (Karlstad University and Uppsala University). The local organisers were Jorryt van Bommel and Arne Engström (Karlstad University).

The programme of MADIF 10 included two plenary lectures by invited speakers, Kenneth Ruthven and Kaye Stacey. As before, MADIF works with a format of full 10 page papers and with short presentations. This year the number of full papers were nine and the short presentations numbered 15. New for this meeting was the symposia format, where three papers around a common theme were presented and discussed. Three symposia were organised this first time. As the research seminars have sustained the idea of offering formats for presentation that enhance feedback and exchange, the paper presentations are organised as discussion sessions based on points raised by an invited reactor. The organising committee would like to express its thanks to the following colleagues for their commitment to the task of being reactors and moderators: Cecilia Kilhamn, Johan Prytz, Anna Wernberg, Thomas Lingefjärd, Anette Jahnke, Olov Viirman, Arne Engström, Anette Bagger, Judy Sayers, Jesper Boesen, Johan Häggström and Ewa Bergqvist.

This volume comprises summaries of the two plenary addresses, nine research reports (papers), three symposia and abstracts for the 15 short presentations. In a rigorous two-step review process for presentation and publication, all papers were peer-reviewed by two or three researchers. Short presentation contributions were reviewed by members of the programme committee. Since 2010, the MADIF Proceedings have been designated scientific level 1 in the Norwegian list of authorised publication channels available at http://dbh.nsd.uib.no/kanaler/.

The editors are grateful to the following colleagues for providing reviews: Allan Tarp, Anna Pansell, Anna-Lena Ekdahl, Anne Birgitte Fyhn, Astrid Pettersson, Birgit Gustafsson, Cecilia Kilhamn, Christina Bauck Jensen, Constantinos Xenofontos, Djamshid Farahani, Frida Wetterstrand, Frode Rønning, Hanna Palmér, Inge Henningsen, Jannika Lindvall, Jenny Tegnefur, Kicki Skog, Kirsti Hemmi, Laura Desimone, Liv-Sissel Grønmo, Madeleine Löwing, Marcus Sundhäll, Maria Bjerneby Häll, Maria Fahlgren, Maria Larsson, Mats Brunström, Morten Blomhøj, Niclas Larson, Olov Viirman, Paul Andrews, Per Nilsson, Petra Svensson Källberg, Robert Gunnarsson, Sofia Öhman, Steve Lerman, Thomas Lingefjärd and Tomas Bergqvist.

The organising committee and the editors would like to express their gratitude to the organisers of *Matematikbiennalen 2016* for financially supporting the seminar. Finally we would like to thank all participants of MADIF 10 for sustaining their engagement in an intense scholarly activity during the seminar with its tight timetable, and for contributing to an open, positive and friendly atmosphere.

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Digital computational tools in school mathematics: ecological, epistemological and existential challenges

Kenneth Ruthven

This paper examines three areas of challenge for the integration of digital computational tools into school mathematics. The ecological challenge involves adapting everyday practices of school mathematics in response to the introduction of such tools. The epistemological challenge involves developing disciplinary and didactical knowledge to guide the use of such tools in school mathematics and the associated evolution of the subject. The existential challenge involves understanding how social representations, values and identities shape the use (and non-use) of such tools.

Responding to the conference theme – *ICT in mathematics education: the future and the realities* – this paper examines three areas of challenge relating to the integration of digital computational tools into school mathematics:

- *Ecological*: adapting the everyday practice of school mathematics to make use of such tools within the operative constraints of time, space and infrastructure.
- *Epistemological*: developing disciplinary and didactical knowledge to inform the use of such tools in school mathematics and the associated evolution of the subject.
- *Existential*: understanding how (collective and individual) representations, values and identities relating to school mathematics mediate the use (and non-use) of such tools.

Ecological challenges

The introduction of digital computational tools into school mathematics involves change in the range of material resources available and sometimes in the physical environment in which teaching and learning take place. These changes bring perturbations to, and adaptations of, established relations between teacher,

Kenneth Ruthven, University of Cambridge

students, subject and tools which I have tried to capture in my *Structuring features of classroom practice framework* (Ruthven, 2009). This framework was developed by taking disparate ideas originally developed to understand such issues prior to the arrival of digital computational tools, and drawing them together in the light of early studies of the introduction of such tools.

The use of digital resources often involves changes in the *working environment* of lessons in terms of room location, physical layout and class organisation, requiring modification of the classroom routines which enable lessons to flow smoothly (Jenson & Rose, 2006). Equally, while new technologies broaden the range of tools and resources available to support school mathematics, they present the challenge of building a coherent *resource system* of compatible elements that function in a complementary manner and which participants are capable of using effectively (Amarel, 1983). Likewise, innovation may call for adaptation of the established repertoire of activity formats that frame the action and interaction of participants during particular types of classroom episode, combining to create prototypical *activity structures* for particular styles of lesson (Leinhardt, Weidman & Hammond, 1987).

Moreover, incorporating new tools and resources into lessons requires teachers to develop their *curriculum script* for a mathematical topic, the cognitive structure which informs their planning of lesson agendas, and enables them to teach in a flexible and responsive way. This structure covers variant expectancies of events and alternative courses of action, forming a loosely ordered model of goals, resources and actions for teaching the topic. It interweaves mathematical ideas to be developed, appropriate topic-related tasks to be undertaken, suitable activity formats to be used, and potential student difficulties to be anticipated (Leinhardt, Putnam, Stein & Baxter, 1991). Finally, teachers operate within a *time economy* in which they seek to optimise the ”rate” at which the physical time available for classroom activity is converted into a ”didactic time” measured in terms of the advance of knowledge (Assude, 2005).

Let me illustrate this framework through the particular case of a teacher developing his teaching practice to make use of dynamic geometry (Ruthven, 2010). In terms of *working environment*, each session started in the teacher's normal classroom and then moved to a nearby computer suite. This movement between rooms allowed the teacher to follow an activity cycle in which working environment was shifted to match changing activity format. Starting sessions in the classroom avoided disrupting established routines for launching lessons, providing an environment more conducive to maintaining student attention ”without the distraction of computers in front of each of them.” In the recently set up computer suite the teacher was still establishing start-up routines with students for opening a workstation, logging on to the school network, using shortcuts to access resources, and maximising the document window. The teacher was also developing shut-down routines. Near the end of each session, he prompted students to save their files and print out their work, reminding them to give their file a name indicating its contents, and to put their name on their document to make it findable amongst output from the shared printer.

In terms of *resource system*, this teacher saw work with dynamic software as complementing the established construction work with classical manual tools which preceded it, by strengthening attention to geometric properties. Nevertheless, he felt that old and new tools lacked congruence, because certain manual techniques appeared to lack computer counterparts. Accordingly, he saw dynamic software as involving different methods and having a distinct function: ”I don't think there's a great deal of connection. I don't think it's a way of teaching constructions, it's a way of exploring the geometry.” The teacher was also concerned that students were spending too much time on cosmetic aspects of presentation. He was trying out a new lesson segment which involved showing students an example illustrating to what degree, and for what purpose, it was legitimate to ”slightly adjust the font and change the colours a little bit, to emphasise the maths, not to make it just look pretty”, so establishing socio-mathematical norms for using the new tool.

In terms of *activity structure*, the teacher's account of his lesson pointed to a combination of activity formats: ”a bit of whole class, a bit of individual work and some exploration”; a structure that the teacher wanted ”to pursue because it was the first time [he]'d done something that involved all those different aspects.” The teacher also highlighted how arrangements had not worked as well as he would have liked in fostering discussion during student activity. He would be giving more thought to how best to organise this. The teacher noted ways in which use of the software helped to structure and support his exchanges with students, creating three-way formats of interaction between student, computer and teacher. Such opportunities arose from helping students to identify and resolve bugs in their dynamic geometry constructions. The use of text-boxes created conditions under which students could be more easily persuaded to revise their written comments.

In terms of *curriculum script*, the teacher reported that he was developing new knowledge of ”unusual” and ”awkward” aspects of software operation liable to ”cause a bit of confusion” amongst students, as well as of how to turn such difficulties to advantage in helping students to develop the target mathematical ideas for this lesson. In the first session of the lesson, after an opening demonstration by the teacher, students themselves used the software to construct a dynamic figure consisting of a triangle and the perpendicular bisectors of its edges, and were then asked to investigate this construction. The teacher was developing strategies for helping students appreciate that concurrence of perpendicular bisectors was geometrically significant, by getting them to drag the dynamic figure: ”I don't think anybody got that without some sort of prompting. It's not that they didn't notice it, but they didn't see it as a significant thing to look for.” In the second session, using the dynamic figure that they had constructed the previous day, students investigated how the position of the point of concurrence of the perpendicular bisectors was affected by dragging vertices to change the shape of the triangle (see figure 1). During this session the teacher asked the class about the position of this ”centre” when the triangle was dragged to become right angled. Afterwards, he commented that he ”was just expecting them to say it was on the line” and that he had not anticipated what a student pointed out: ”I don’t know why it hadn’t occurred to me, but it wasn’t something I’d focused on in terms of the learning idea, but the point would actually be on the mid point.” One can reasonably infer that the teacher’s curriculum script developed to encompass this new variant as a direct result of this episode.

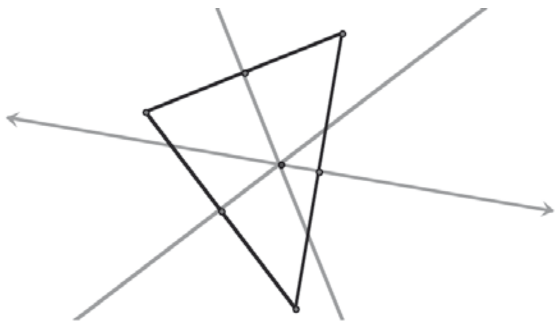

Figure 1. *Example of a dynamic figure under investigation in the lesson*

In terms of *time economy*, this teacher linked his overall management of time to key stages of investigation – ”the process of exploring something, then discussing it in a quite focused way as a group, and then writing it up” – in which students moved from being ”vaguely aware of different properties” to being able to ”actually write down what they think they’ve learned.” Because he viewed the software as a way of engaging students in disciplined interaction with a geometric system, he was willing to spend time to make them aware of the construction process underlying dynamic figures by ”actually putting it together in front of the students so they can see where it’s coming from.” Equally, he was willing to invest time in having students learn to use the software.

This example of a teacher’s developing craft knowledge relating to working environment, resource system, activity structure, curriculum script and time economy conveys the extent of the professional learning involved in adapting everyday teaching practice to the changing ecology created by the introduction to school mathematics of a digital mathematical tool such as dynamic geometry.

Epistemological challenges

Many types of digital computational tool are still at a relatively early stage in their evolution, with significant differences of design between alternative tools of similar type, and between successive generations of a particular tool. This adds to the complexity of establishing stable mathematical didactical analyses. For example, Mackrell (2011) found considerable diversity in basic features of the most commonly used dynamic geometry packages:

- Different repertoires of tools and organisation of them.
- Different styles of interface and modes of interaction.
- Different and inconsistent order of selecting action and object.
- Differing modes of behaviour of figures under dragging.

She comments that ”This diversity is an indication that creating [a] program is not simply a matter of representing the conventions of static Euclidean geometry on a screen, but is dependent on the epistemology of the designer and is influenced by both cultural conventions and pedagogical considerations” (p. 384). Mackrell also comments on the limited volume of research on the impact of such design decisions, even for well-known ones such as selection order and the draggability of objects.

Moreover, the mathematical representations and actions provided by digital computational tools may diverge in important respects from those associated with traditional written inscription. This calls for mathematical didactical analysis to establish a coherent intellectual framework covering the digital and the traditional, and to establish appropriate curricular sequences. For example, the idea of dragging has developed in ways quite unanticipated when the first dynamic geometry software was created. Arzarello et al. (2002) have identified a wide variety of ways in which dragging may be used with dynamic figures, including:

- Wandering dragging: of points without a plan in order to discover configurations or regularities in the drawing.
- Bound dragging: of a point already linked to an object.
- Guided dragging: of the basic points of a drawing in order to give it a particular shape.
- Dummy locus dragging: of a basic point so that the drawing keeps a property.
- Line dragging: drawing new points along a line in order to keep the regularity of the figure.
- Linked dragging: of a point attached to an object.

But we still await development of a full mathematical theorisation of dragging. Equally, little of the didactical analysis necessary to incorporate dragging into the curriculum and to underpin a systematic development of curricular sequences has yet been undertaken.

Such curricular sequences must acknowledge the expansion of mathematical concepts and techniques which digital computational tools make available. While it is relatively straightforward to bring such tools to bear on familiar tasks, ultimately a renewed curriculum must incorporate tasks which would be inconceivable without the mediation of digital computational tools, and this, of course, calls for a corresponding shift in mathematical thinking. Laborde (2001), reflecting on a multi-year project working with teachers, has identified a progression in types of curricular scenario employing dynamic software (which I exemplify here in relation to the geometrical topic already discussed):

- Facilitates material aspects of familiar task: e.g. construction of a diagram consisting of a triangle and its perpendicular bisectors.
- Assists mathematical analysis of a familiar task: e.g. through dragging the triangle to identify the concurrence of perpendicular bisectors as an invariant property.
- Substantively modifies a familiar task: e.g. dragging the triangle to identify a variable characteristic which correlates with the internal or external positioning of the circumcentre.
- Creates a task which could not be posed without dynamic software: e.g. a task in which three circles have been constructed with a common free centre, each circle passing through a different vertex of the triangle; dragging of the ”free” centre is then used to identify the conditions under which two or all three of the circles coincide (see figure 2).

But tools differ in complexity. Dynamic geometry systems are relatively complex tools which radically augment available mathematical representations and actions, and which have not yet achieved stability in design. By contrast, the arithmetic calculator is much less complex, employs broadly familiar mathematical representations and actions, and has achieved relative stability in design. Its use in primary mathematics was the subject of extensive developmental work through the ”calculator aware” number project which influenced the English National Curriculum established in 1989 (Shuard et al., 1992). In particular, this led to the inclusion in that curriculum of a detailed section on calculator methods – alongside more extensive sections on mental and written methods – which evolved over time.

Official guidance recognised that, despite its congruence with established mathematical representations and actions, the introduction of the arithmetical

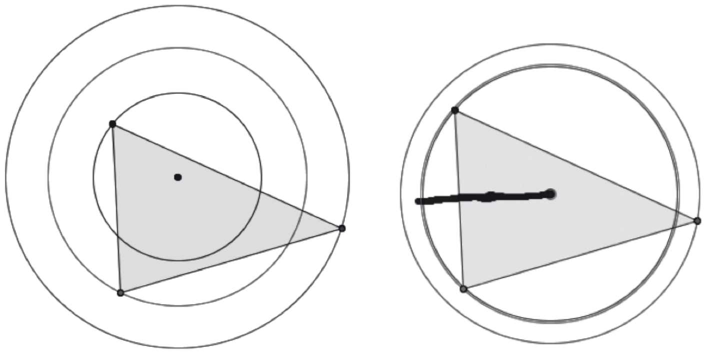

Figure 2. *Construction for investigating where to position a common centre so that circles, each passing through one triangle vertex, coincide*

calculator to primary school mathematics had a number of implications for curricular sequences. In particular, the availability of a calculator made it possible for children to tackle problems using real data relating to familiar situations from an early stage; use of, and experimentation with, the calculator led to children encountering negative numbers and decimal fractions much earlier than in the traditional curriculum; the ease of computation with a calculator made experimental methods of problem solving based on trial and improvement much more feasible.

However, the English National Curriculum has never specified any standard calculator methods of computation to mirror standard written methods. In the cases of addition, subtraction and multiplication, of course, the relatively straightforward way in which such operations are performed on the calculator means that there can hardly be said to be a distinctive calculator method as such. However, the case of division is less straightforward, because the calculator carries out a particular form of division, meaning that it can be necessary to interpret the calculator result and translate it into an alternative form. Had the curriculum made explicit the need for a calculator-based method of quotient and remainder division, this would have provided a publicly visible capstone for the ”calculator aware” curriculum. Such a capstone could stand alongside – if not replace – the long division algorithm, the culmination of the traditional written arithmetic curriculum, which, for many members of the public, is a totemic mathematical achievement.

Existential challenges

The introduction of digital computational tools has a potential to modify and to be perceived to call into question certain established features of school mathematics. To the extent that such features are highly valued, particularly by powerful and influential groups, the introduction of these tools, or development of their use beyond a certain point, is likely to encounter reluctance or more active resistance. This process is mediated by the social representations in circulation: the simplifying models through which people make sense of a new and unfamiliar phenomenon by relating it to more established and familiar ones.

It is the calculator which has become the popular archetype around which prevalent social representations of digital computational tools in school mathematics are formed. In England, at least, this is reflected in a shift in policy that has taken place at all levels of schooling towards a curriculum that is more ”calculator beware” than ”calculator aware”. In 2011, when the English government announced a review of the National Curriculum, this extract from a press release by one of the politicians who was to serve as schools minister during the period of the review conveys the government’s sentiments on this matter:

> We should ensure that schools equip children with the mathematical basics that allow them to succeed in life. We are in danger of producing a ”Sat-Nav” generation of students overly reliant on technology. (Truss, 2011)

Discussion on an online comment board – from *The Guardian*, a newspaper with a broadly liberal readership – provided me with an opportunity to gauge popular opinion on this matter and to analyse the associated social representations (Ruthven, 2013). Many of the comments depict the use of calculators by pupils as antagonistic to thought and subversive of intelligence:

> One of the most important things that a child learns is the ability to think. If you give them a tool that discourages that at such a young age, that aspect of their thinking will be stunted.

> A child’s mind needs exercise just as their body does.

> Nothing clever about using a calculator to work out numbers.

Some comments portray use of calculators not only as developmentally debilitating but as a morally iniquitous avoidance of effort:

> Using a calculator […] rots the brain, not to mention the poor ethic it instils […] if they don’t work out the answers with hard graft.

> Going straight for the answer is the easy, cheap and wrong way to go.

Where contributions concede that using a calculator does involve a degree of expertise, this tends to be presented as distinct from mathematics itself:

> Learning to work a calculator is only learning to work a calculator, not learning how to do maths.

A common suggestion is that access to calculators should be granted pupils only once they have become confident with number and proficient in mental or written calculation:

> They should learn how basic arithmetic works first, which means doing it either in their head or on paper.

Some comments are salutary in showing how opposition to calculators is embedded in contributors' sense of personal worth, grounded in their own educational experiences. One such identity narrative from a contributor conveys a sense of personal accomplishment associated with mastery of mental and written calculation, expressed in a continuing proud refusal of the calculator:

> I learned arithmetic the old-fashioned way, using a sums book, following the methods demonstrated by the teacher on the blackboard. By six, I could add and subtract up to a hundred, by eight I had long division and multiplication, and all the tables to ten […] My mathematical skills took me all the way through A level into degree-level statistics, and then a (boring) first job in Health Service data analysis. I have never owned a stand-alone calculator, and I don't use the one in MacOs X.

Another (atypical) identity narrative conjures up a very different type of personal history, offering a sense of how, for some pupils at least, calculators serve as a catalyst for developing interest and capability with numbers:

> I am really good at mental arithmetic, but as a child abhorred rote learning of times tables, couldn't see the point as I could work them out in an instant. It almost alienated me completely from maths, luckily playing with calculators […] rekindled my interest in number games. So when I was older and scientific calculators starting coming in […] I used to play with it, especially the functions that worked out means and standard deviations. That set me up well for the types of maths I used in later life, inferential statistics.

This narrative indicates that there are some threads of popular opinion which are more positive about calculator use in school mathematics. Nevertheless, it seems that the currently dominant strands of popular thought tend to devalorise the use of calculators in particular, and of digital computational tools more generally, through the following polarised associations:

- Cognitive self-sufficiency: thinking ”independent” of digital tools versus unthinking ”dependence” or ”over-reliance” on such tools.
- Mathematical essence: ”purely” mathematical mental/written methods versus (wholly/partially) ”non-mathematical” use of digital tools.

– Moral virtue: ”effortful” use of ”rigorous” mental/written methods versus ”lazy” recourse to ”slipshod” use of digital tools.

– Epistemic value: use of mental/written methods taken as exercising intelligence and developing understanding versus use of digital tools taken as doing neither.

Transforming such representations, many of which are shared, at least in part, by educators themselves, represents a considerable challenge for the field.

Conclusion

In the light of these – still not well understood – ecological, epistemological and existential dimensions, we should not be surprised at limited progress to date in integrating digital computational tools into school mathematics. At the same time, this paper has tried to show how research has started to give us a better understanding of these issues, and has the potential to develop new knowledge which can help to tackle these challenges.

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Researching the use of ICT to teach mathematics: the case of mathematically able software

Kaye Stacey

The purpose of this paper is to present a broad survey of research questions, methods, and a few findings from over twenty years of research with various colleagues centred around the University of Melbourne and to suggest important issues for research. The paper will focus on questions specifically related to mathematics teaching and to the use of what we call ”mathematically-able software”. This is only a part of the ICT that mathematics teachers use, and indeed our projects extend beyond this focus (see for example Pierce & Stacey 2011; Price, Stacey, Steinle & Gvozdenko 2013; Stacey & Wiliam 2013). However, the focus on mathematically-able software is a critical one because this is the software that is most challenging to mathematics. This is responsible for ICT being one of what I see as the two major drivers for change in mathematics curriculum in our time (the other being the growth in the percentage of students attending secondary schooling around the world).

Because ICT is now such a major force in mathematics education, it is evident that studies of ICT in mathematics encompass very many aspects of curriculum, teaching and assessment. Consequently, they must draw on a diverse range of questions and theories, all motivated by the opportunities ICT offers to improve mathematics outcomes for students.

ICT in Australian schools

The work that I report has been carried out in the context of Australian schools. As a general rule, Australian people like the idea of using up-to-date technology and this is reflected by national expectation, supported in the Australian Curriculum and in government funding policies, that ICT should be used in schools (Government of Australia, 2013). In recent years, the major thrust has been to use ICT across the school in all subjects, and so this has strongly promoted the use of internet resources for research, digital textbooks and other learning resources, software for student presentations, collaboration tools, and learning management systems. ICT of this nature is principally a communications infrastructure for schools. This paper focusses on ICT forming a computational infrastructure for school mathematics.

Kaye Stacey, University of Melbourne

Within mathematics, Australian curricula and examinations (the 8 states and territories have somewhat separate systems) have used four function calculators in primary schools and scientific calculators in secondary schools since the early 1980s. Secondary school mathematics now generally uses complex calculators, including in university entrance examinations. In my state of Victoria, graphics calculators have been widely used and permitted in examinations since about 1995 and CAS calculators have been phased in for Year 12 examinations progressively since 2002. More precise details as well as a summary of research into these initiatives are reported in Stacey (2016) and in many of the references to this article (e.g. Leigh-Lancaster, 2010). Teachers are encouraged to use software such as spreadsheets and dynamic geometry in class. Of course, there remains some difficulties with resources (e.g. it is often a very difficult task to set up a data projector in a room) and not all teachers know about available resources or have up-to-date skills.

Mathematically-able software (MAS)

As noted above, this paper focusses on research into the use of mathematically-able software. These are open tools, where the user (generally in this case a student) inputs ”questions” in mathematical language to which the software provides answers. The classic MAS tools can be used in life outside school. Examples are calculators of all sorts, computer algebra systems (Mathematica, Maple, etc.) abbreviated here to CAS, statistics packages, and spreadsheets (e.g. Excel). However, we also include some software unlikely to be used beyond school such as dynamic geometry (e.g. Cabri, Geometer’s Sketchpad) and some applets with an open mathematical capability even if the topic is limited such as some of those from the National Library of Virtual Manipulatives from the Utah State University. Unlike educational software that directs what the user will do (for example, by presenting a series of questions to answer, or actions to undertake in a game), the teacher or student decides what to do with the MAS.

Because they have a role beyond school, MAS challenges the goals of education and what techniques are taught to students, whilst providing opportunities for learning. Much of our Melbourne research has been inspired by these challenges and opportunities. Since 1990, we have conducted research projects on most of these software tools.

Structuring the research program

The diagram in figure 1 shows that we have structured our research program on MAS around three central themes. First, as noted above, the widespread accessibility of MAS outside school presents a challenge to the content and the goals of the school curriculum. Second, a major concern for educational systems in adopting MAS in schools is related to assessment. In our state, this concern has principally been directed at the end-of-school examinations for Year 12 students.These examinations are set and marked by the state authority. They are high-stakes for students and teachers, because the certification relates to school completion and performance is the most important factor in selection into university courses. This has been the major driver of adoption of CAS in secondary schools often from Year 9 up. The third theme relates to the "pedagogical opportunities" that are supported when MAS is available for teachers and/or students. Curriculum change and assessment change are principally influenced by what we call calculation use of MAS (e.g. finding an integral, multiplying two multi-digit numbers, inverting a matrix, calculating a regression equation, or graphing a function). However, since the very beginning of affordable digital technology, it has been recognised that it can also be used in order to improve learning of mathematics. We call this "pedagogical use". For example, Etlinger (1974) noted how a calculator could be used to illustrate the subproducts that are involved in the long multiplication algorithm. At the time, very long multiplication was still regarded as a valuable skill because the calculator could only display about 6 digits. He also noted the pedagogical benefits of discussing anomalies such as $(1 \div 3) \times 3 = 0.9999999$ which occurred on the calculators of the day. Behind all these themes of use of MAS is the practical need to investigate questions of teacher professional development.

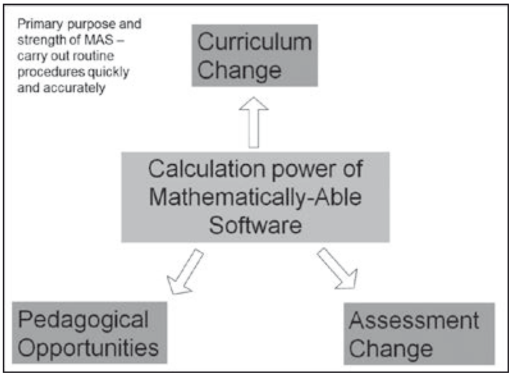

Figure 1. *Three themes of research on the use of MAS software*

In the next sections, I will examine each of the three parts of the research project in turn briefly and suggest areas where research is still needed. In these sections, it becomes evident that each issue is multi-faceted and so research into each inevitably draws on a wide range of research methods and frameworks. The relevance to ICT in mathematics teaching cuts across many concerns and so it cannot be pigeonholed as just one strand of research.

Curriculum change – content and goals

It has been clear for nearly five decades that the advent of digital technology has profound implications for the mathematics curriculum – its content and its goals. Of course, different people assess these implications in different ways. Whereas there appears to be almost universal agreement that ICT should be used ”to teach better” (so that students to have better understanding, more confidence etc.), there is less agreement about what topics are now obsolete or of low priority, what new topics should replace them, and how the presence of ICT should change the goals of mathematics (see for example Ball & Stacey, 2001, 2005). For myself, I want the process of gradual pruning of obsolete topics to continue (it has been going on in Australia since at least the 1980s). I feel that considerably more attention should be given to new topics and especially to more consistently shifting to ICT-aware methods (e.g. using spreadsheets). Most of all, I would like ICT to assist us to more strongly emphasise the, often stated, goal of students becoming better problem solvers, especially being better able to formulate real world problems mathematically, better able to conduct mathematical investigations and creating a school mathematics that is less dominated by routine procedural work.

Although there is important work on each of these goals already, the questions below are still in need of research:

(a) Can we better illuminate the links between achieving competent use and full understanding of a topic and practising its procedures? Is the strength of the link different for different topics? For example, I hear no-one now decrying serious consequences of the lack of practice of the pen-and-paper algorithm for calculating square roots, which was widely taught until about 50 years ago; and there appears to be no real concern in the statistics education literature about the use of statistics packages. On the other hand, there seem to be endless debates on arithmetic and algebra. Reading Etlinger (1974) puts this is context.

(b) Mathematics with ICT is a different subject – can this be elaborated? What is the nature of that mathematics? How does using MAS change students' understanding of a topic? There is substantial knowledge of this only in a few areas (e.g. graphing, dynamic geometry dragging).

(c) What curriculum will really equip students for the ICT future, and how can large education systems make the changes needed to get there?

(d) Are there new MAS that will impact on school curricula soon? For example, when will we have a tool which makes three-dimensional mathematics very feasible and what will we do with it?

Research on these questions, and research on curriculum changes that has already occurred requires a range of methods and frameworks, including drawing on theoretical mathematical and document analysis, theories of adoption of innovation, studies of attitudes, and empirical studies of learning.

Assessment change

As noted above, most of our research on assessment change has largely been related to accommodating the calculation power of MAS, especially that of graphics calculators and CAS calculators. A diverse group of researchers has examined the evolving practices and data from Victorian Certificate of Education through the introduction of graphics calculators (1995+) and CAS calculators (2002+) and now computer software. The findings are outlined in Stacey (2016) and a wide range of references are given there. This is summative assessment in a high stakes environment. School educational authorities in several parts of the world beyond Australia (e.g. Denmark, Scotland) and many university departments have also faced this issue. However, the question of designing assessment for a technologically-rich environment is broader. For example, the OECD's PISA 2012 survey (OECD 2013) had an optional computer-based assessment of mathematics (CBAM) that included some items which assessed unchanged mathematics in a way that was enhanced by the computer-based format and other items where students could use some computational power of the computer. Within the Assessment Change strand, research methods include empirical studies (natural and designed experiments, comparisons and surveys), document analysis and mathematical-didactic analysis, and theoretical studies especially related to values. Many studies have a local focus with international input and some generalizability.

We have organised our research and development work on assessment under three guiding principles (Stacey & Flynn, 2007). The first is the ”Mathematics principle”, referring to the imperative to assess mathematics that is important for students to learn. Much research on this has been reported (see for example Flynn & McCrae, 2001; Stacey & Wiliam, 2013) so space here permits only a few sample results. Our work under this principle has established that creating good assessment items for the new computational environment requires a new set of skills, because many items that were formerly testing significant mathematics now test something different. Routine, procedural, questions are the most affected because students can generally just ”type in” the question, so it is tempting to consider removing these relatively easy items from examinations in order to make space for items where students can use the technology to support more substantial problem solving. However, care has to be taken with this approach to ensure that examinations with technology do not become inappropriately difficult for students.

Along with others (e.g. Brown, 2010) we have found that evolution away from a largely procedurally-based examination is very slow with only weak drivers for change. Teachers mostly want to ”teach better” rather than to extend the expectations for students, so there is a tendency for CAS to be used to compensate for weaker skills rather than amplify what students can do. Moreover, in common with many other countries, our examination system now includes an ”ICT-forbidden” component. This political compromise removes pressure to update the curriculum.

The second principle for assessment is the ”Learning principle” – giving consideration to assessment actions that promote good effects in the classroom from both the teacher’s and the students’ perspectives. Assessment in our settings is the most powerful driver of what happens in classrooms, so it is important that assessment design promotes good classroom practices. For example, it seems important that assessment should, as far as possible assess mathematical thinking rather than technical proficiency with button pushing. Perhaps the most important work related to this principle has been to compare the performance of students who learned mathematics with CAS to those who learned with a graphics calculator. Students were permitted to use the technology they learned with, when answering the examination questions. A series of studies from 2006 to 2009 demonstrated that the students using CAS always performed slightly better than those with graphics calculators. This was the case in the components with and without ICT, and when controlled for factors related to general ability. In general, a similar percentage of students in both groups did very well, but fewer CAS students did poorly, and the average score of the middle students was slightly higher. The results are summarised in detail by Stacey (2016) along with several possible explanations.

A third series of studies have examined the ”Equity principle” – ensuring that the assessment is fair to all students, regardless of the hardware or software which they use. These studies have included investigations into: the effect of allowing technology on groups of students (e.g. low socio-economic status, girls); whether some brands and models confer an advantage (and procedures to ensure that this does not happen); empirical studies of effects on individual questions; and comparison of the use of computers with hand held devices. An important observation is that the ICT skills of teachers seems to be an important equity factor for their students, and an important design question is to consider what conditions would enable ICT to open up opportunities for more students to participate in advanced mathematics. The studies are also summarised in Stacey (2016).

Pedagogical Opportunities

Our work on the pedagogical opportunities offered by ICT has been structured by the pedagogical opportunities map, shown in figure 2 (Pierce & Stacey, 2009;

2010). We have used the map to guide our research program, and as a tool for tracking changes in teachers' practice in our studies on teacher-learning. We also use it in our work with in-service teachers to present them with a smorgasbord of ways in which they may choose to enrich their practice. It is important to note that all the opportunities arise because MAS carries out computation quickly and accurately and displays can be shared.

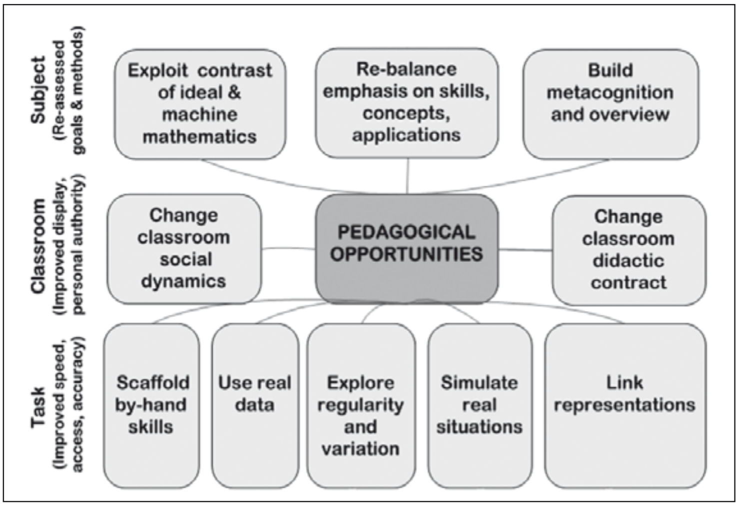

Figure 2. *The map of pedagogical opportunities for MAS*

There are three levels in the map of pedagogical opportunities. ICT can be used to support new types of tasks and to facilitate student work on tasks that were previously impractical. The bottom row, the ”Task” level, identifies general ways in which teachers can use MAS to enhance teaching. The first box there draws attention to ICT being used to support students' learning of standard pen-and-paper skills or for developing concepts. For example, CAS can be used to demonstrate the usefulness of equation solving before students can solve equations independently, and later to check solutions to equations, or share part of the cognitive load as they learn the new procedures. Other boxes highlight how computational power supports students in their exploration of situations as they look for patterns, study the impact of variations, or use simulations. A much heralded possibility in the research literature is to give students very easy access to multiple representations of mathematical phenomena. This is in the rightmost box and is discussed below.

The second row highlights two ways in which teachers may use ICT, in particular MAS at the classroom level. MAS provides an opportunity to change the social dynamics of the classroom especially through the use of shared screens and easy display of student work for enhanced discussion and collaboration. There is also an opportunity to change the didactic contract, which is that part of the complex set of relationships of obligations between teacher and student in a classroom that is specific to the mathematical knowledge. The opportunities to change the didactic contract can arise in several ways. For example, ICT/MAS supports an ”explosion of methods”; the number of methods available to solve problems increases sharply beyond the number of methods that are practical in a pen-and-paper environment. Students therefore may have more to contribute to a class discussion, and being able to check their work with CAS can make them more confident to make these contributions (Pierce, Stacey & Wander, 2010). On the other hand, introducing MAS can lead to a mismatch between students' and teachers' understanding of the didactical status of knowledge within classrooms, part of the didactic contract. For example, the study by Pierce, et al. (2010) found 77 % of students but only 1 of 6 teachers identified learning to use ICT as a main point of a certain lesson that they all taught. In the classroom some students wanted to have very small details of their ICT use confirmed by the teachers, behaviour that teachers saw as wasting time and drawing attention away from mathematics. This was a consequence of the break in the didactic contract and the teachers were later able to address it.

The third ”Subject” level of pedagogical opportunities points to ways in which mathematics as a subject to learn can be altered by using MAS. For example, a very early research study (Heid, 1988) demonstrated how a calculus course could put primary emphasis on concepts and applications, rather than giving priority to the initial development of skills for differentiation. We have worked with teachers who use CAS to provide an overview of the topic and where it leads to enhance students' understanding of why they are learning mathematics – what one of our teacher colleagues calls ”teaching the ends of a topic” (Garner, McNamara & Moya, 2003). And as Etlinger (1974) demonstrated right at the beginning of the technological revolution with his suggested investigation of why calculating 1/3 x 3 did not give the exact answer of 1, there are many possibilities to highlight mathematical thinking by observing the limitations of technology or the contrast between ideal mathematics in the head and enacted on a machine.

Investigating how to teach with multiple representations

Our research on pedagogical opportunities has used the pedagogical map of figure 2 in two ways. We have used it as a tool to map teachers' practice (see for example Pierce & Stacey, 2010) and we have also used it as the basis of our program, to better understand how to teach with ICT. In one instance we were invited to conduct a two year professional development program at a school that was introducing use of Texas Instruments TI-Nspire from Year 9 to 12 (Pierce & Stacey, 2009; 2013). TI-Nspire CAS software has symbolic algebra (e.g. solve an algebraic equation, including with parameters); graphing, dynamic geometry, tables of values, statistical functions and a document facility for multi-page investigations. All of the capabilities are linked.

At one time, we were invited to help the teachers design a ”cap-stone” lesson for a unit on quadratic functions for Year 10 students, taking about 100 minutes of class time. Teachers wanted to learn about the pedagogical opportunities and also the capabilities of the device. The students had handheld machines and the teacher used the parallel computer display on an electronic white board. With the teachers we designed a lesson, which was taught multiple times in a lesson-study context. The lesson was rich in representations of the central structure: geometric, symbolic, graphic. There is strong support for using multiple representations in the research literature, especially because conceptual knowledge has rich connections, because thinking mathematically involves exploring mathematical ideas from several perspectives, and because seeing a mathematical structure in different representations highlights different features. The lesson is described by Wander and Pierce (2009) and student material is available from RITEMATHS website (not dated). A large body of research on graphics calculators shows benefit of working with symbol-graph-(table) representations of functions. TI-Nspire CAS software provided many more possibilities. The lesson poses a series of questions about a fish shaped sign of fixed length, to be installed above Marina’s fish shop (see figure 3). One main question relates to minimising the area of the fish by adjusting the length of the body of the fish. These students had not studied calculus. Figure 3 shows possible fish shapes, the formula for area as a function of body length, the graph of the area

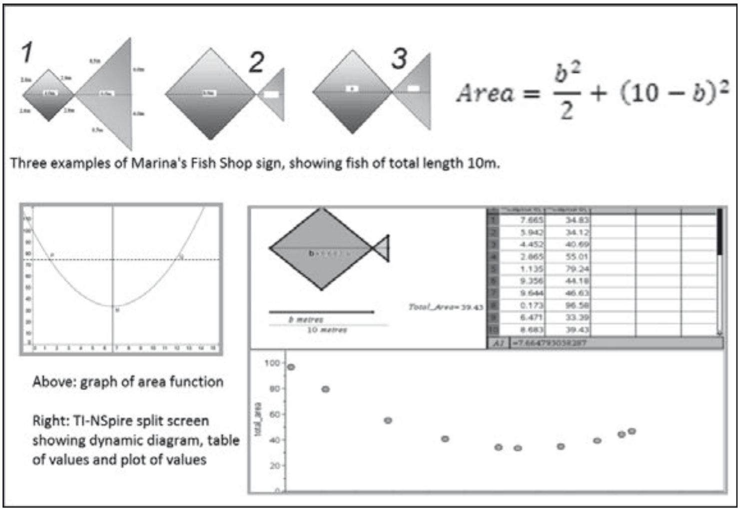

Figure 3. *Images from Marina’s fish shop lesson*

function and the triple screen from TI-Nspire showing the manipulable fish shape, the table of values and the plotted points used for quadratic regression.

The design research highlighted four aspects of teaching with multiple representations (Pierce, Stacey, Wander & Ball, 2011). The first was to highlight the need to avoid too much cognitive load when working with multiple representations. The lesson had verbal descriptions, pictures, a dynamic diagram with and without measurements, and later the symbols and the two sorts of graphs. Too many! This is a strong message from the HCI literature that has not been prominent in the mathematics education literature. However, as MAS technology adds functionality, it will become more prominent. Prepared screens (e.g. a movable fish in dynamic geometry) save time and keep focus on mathematics rather than ICT but effort is required for students to appreciate even simplest screens and small changes in representation. It is easy to underestimate this.

A second issue concerned student motivation. We began with the belief that examining the same phenomenon in different representations would lead to good learning. Whilst this may be true when the representations are new to students, in our first iteration of the lesson, students became bored when they had to find the minimum value empirically and from the graph. We had more success if each representation was used to inform a different part of the total investigation.

Another issue was to select the focus of the lesson and discard other possibilities. This arose because so many functionalities were available. The ”explosion of methods” meant that there were many ways of using the technology to solve a problem such as finding the minimum area. However, this created the need to choose carefully what we intended to be the focus of the lesson. Because teaching operates on a time economy, many good opportunities have to be passed up in any one lesson.

A fourth issue highlighted differences between ideal mathematics and mathematics within a device. In mathematics-in-the-head, there can be one variable ”the body length of the fish” which is used in drawing fish, in the algebraic formula, to label the columns of the table of values, to label the slider length on the dynamic diagram, and to report the quadratic regression. However, mathematics within a device must keep all of these occurrences of the body-length variable separate, with different names. Consultations with programming experts confirmed that this difference is inherent and not readily smoothed over. With multiple representations, variable naming can cause a semiotic storm!

Learning to teach with technology

Another section of our work has related to learning to teach with MAS technology. This change has required major investment by educational authorities and schools to help teachers develop the new technical and pedagogical skills that they need. The mandated change to assessment at Year 12 has been the major driver of the change, but schools have also been keen that students in Years 10 and 11 use the same device as they will use in the final examinations at Year 12. In several of our studies, we have found that learning to teach with MAS can be very hard for many of the teachers who are outside the small percentage of self-motivated early adopters. For example, even after 2 years supported use, some teachers in the professional development program described above, were still learning to use MAS to a level they see as adequate for teaching. Their confidence was still growing, along with their trouble shooting skills, and recognition of possibilities. Other studies have wider data, e.g. Pierce and Ball (2009) and Pierce and Stacey (2004). Most teachers find it easiest to use MAS just to calculate, rather than as pedagogical tool. The pedagogical opportunity of exploring regularities seems to be the first to be adopted. Time for learning is an important barrier for teachers, and regular updates of technology make the task a continuous challenge.

Reflections on ICT-related research

Using ICT, even considering only mathematically-able software, affects all aspects of teaching mathematics: the content and goals, the assessment, the classroom environment and the tasks on which students work. Taking advantage of the new opportunities and addressing the new challenges is a long term journey for mathematics teaching. It began at the start of the digital era when four-function calculators became sufficiently portable and affordable that nearly everyone could own one, simultaneously displacing the log tables and slide rules used by professionals. Since then, the unabated increase in mathematical and other capabilities of ICT ensures that there is no steady state just around the corner. Since all aspects of mathematics teaching are affected, the research methods, research questions and insight-delivering theoretical perspectives required are extremely diverse. There are basic insights related to technology, such as the notion of distributed cognition (Pea, 1987) that stresses the fundamental importance of studying the capabilities of the person plus the tool as one unit. Beyond such basic insights, research into ICT stretches across the concerns of mathematics education.

A particular challenge for ICT-related researchers is timeliness. Research is most useful if it has something to say to practitioners when it is needed, and this demands working at the technological forefront. However, it is not possible to conduct extensive research related to a technology which is not yet easy to use in schools. Studies that try to work too far ahead often deflect towards issues of implementation (such as access to equipment which may quickly or slowly change) and these issues can overwhelm the mathematical, didactic and pedagogical findings that are required to guide practice in the longer term.

A second aspect for ICT-related researchers arises from the necessity to use devices that are engineered and marketed in particular ways. Again, the best research for mathematics education will focus on the fundamental characteristics of the device for teaching mathematics. This also means that concepts from engineering and product design such as work on user-experience and the adoption of innovations, may be profitably brought into mathematics education research more than is currently seen.

In regard to the three aspects of broad aspects of research on the impact of MAS technologies, it seems that there is still much to be done to build expertise in lesson design which captures the pedagogical opportunities of MAS. For assessment change, especially in regard to summative examinations, a large body of experience has accumulated in certain geographically-diverse countries related especially to algebra and functions, and statistics (see for example Drijvers, 2009). For other settings, there is little research specifically related to the use of MAS – most assessment research (see for example Stacey & Wiliam, 2010) relates to using the capacity of ICT for item presentation, selection, scoring and reporting etc. rather than MAS issues. The role of MAS in assessment is particularly important because it can drive both teachers' and students' actions. Behind all of this sits judgements and values about the way in which MAS can and should change the content and goals of school mathematics. I think that a reasonable judgement is that there has generally been only very slow adaptation to our new technological environment. With all of these exciting but challenging developments, I judge that the elusive goals of deeper and better mathematics remain elusive, but maybe not quite as elusive now we do have these technologies to help us.

Acknowledgement

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Swedish students in upper secondary school solving algebraic tasks – What obstacles can be observed?

Birgit Gustafsson

To understand more about students' difficulties when doing algebraic problem solving Duval's semiotic theory and a mathematical modelling cycle are used to identify what obstacles can be observed. The results show that when the students have to perform transformations between two different semiotic representation systems – a conversion – the obstacles get visible.

The focus in this paper is to investigate what obstacles can be observed while students are doing algebraic problem solving, and how these obstacles can be characterized. The students are in the first year of the social science- and natural science program in three different upper secondary schools in Sweden. Algebra is a quite new area for the students at this level and it is known as an abstract and problematic area A large body of research over a long period of time has documented the learners' difficulties both with the use of variables and the understanding of the nature of algebra as such (see e.g. Puig & Rojano, 2010; Stacey & Chick, 2010). Algebra is seen by many as a very abstract part of mathematics and as it is often taught, it has been characterized as "a watershed for most people" (Mason, 1996, p. 65).

However, algebra is versatile. When algebraic symbols are applied, it depends on the specific problem what one sees in them and what one is able to see. Drijvers (2003) suggests that problem solving in algebra is more difficult than problem solving in other areas. Among other things, this may depend on the abstract level at which algebraic problems are to be solved and the algebraic language with its specific symbols.

The use of letters in mathematics has been pointed out as a difficulty for students. In mathematics, letters are used in equations, in formulas, in functions, for generalization of pattern and so on (Drouhard & Teppo, 2010), and it is essential that students understand that letters always stand for numbers. It has been known for a long time that understanding the role of letters in algebra is very difficult for many students (see e.g. Küchemann, 1981; MacGregor & Stacey, 1997).

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The overall aim of this study is to learn more about upper secondary school students' difficulties while doing mathematics and how they interpret the mathematical content. For this particular paper I am going to address the following research question.

> What obstacles can be observed when students in upper secondary school discuss algebraic problem solving in groups and how can these obstacles be characterized?

To answer the question a theoretical framework consisting of two parts will be applied. The identification of obstacles will be analysed using a modelling cycle (Lester & Kehle, 2003) to find where in the cycle the obstacles appear. To further characterize the obstacles, theory about transformations between semiotic representations (Duval, 2006) will be used.

Theoretical framework

Transformations between semiotic representations

Mathematical knowledge is a special kind of knowledge. It is not like other sciences because mathematical concepts or objects are abstract. Therefore there is no direct access to mathematical objects. To gain access to these objects the only way is by using semiotic representations. However, the signs have no meaning of their own and depending on whom you ask, you may get different answers, depending on the person's conceptions and experiences of the particular object (Duval, 2006). Duval proclaims that "the leading role of signs is not to stand for mathematical objects, but to *provide the capacity of substituting some signs for others*" (p. 106). This is what Duval refers to as *transformation* and he describes two different types of transformations, *treatments* and *conversions*. Treatments are transformations within one semiotic system, such as rephrasing a sentence or solving an equation. Conversion is a transformation that involves a change of semiotic system but maintaining the same conceptual reference, such as going from an algebraic to a graphic representation of e.g. a function. Duval uses the word *register* to denote a semiotic system that permits a transformation of representations (p. 111). Duval claims that changing representation register, i.e. performing a conversion, is the most challenging transformation for students.

Duval groups registers into *monofunctional* and *multifunctional*. A monofunctional register involves mathematical processes, which mostly take the form of algorithms (e.g. algebraic formulas). A multifunctional register consists of processes that cannot be made into algorithms (e.g. natural language) but involves other types of cognitive functions such as communication, awareness and imagination. Furthermore he distinguishes between *discursive* and *non-discursive* registers where the former type is of the kind that e.g. involves statements of relations or properties, or statements about inference or computation, and the latter type consists of e.g. figures, graphs and diagrams. This gives four types of registers and transformations can take place between (conversions) and within (treatments) all four types (see Duval, 2006, p. 110).

In learning mathematics, the cognitive complexity of comprehension is touched through various kinds of conversions, more than through treatments. For example in a conversion task, when the roles of source register and target register are inverted, the problem can be changed for the students, and then they often fail. Many misunderstandings lie in the cognitive complexity of conversion and the change of representation. In a conversion a rephrasing can change the complexity of the situation. A conversion where the transformation from one register to another can be done by translating ”sign by sign” turns out to be easier to handle than one where this is not the case. Duval refers to these types as *congruent* transformations and *non-congruent* transformations, respectively (2006, pp. 112–113).

The mathematical modelling cycle

Problem solving involves mathematical modelling. To solve a problem students have to first simplify the complex settings (Lester & Kehle, 2003). That involves interactive use of a variety of different mathematical representations.

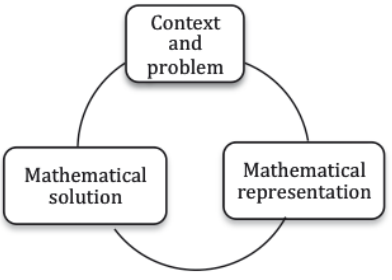

Figure 1. *A mathematical modelling cycle (Based on Lester & Kehle, 2003, p. 98)*

According to Lester and Kehle (2003), the problem solving process begins with a translation of the *problem* posed in terms of reality, into abstract mathematical terms. This involves making a decision about what could be omitted, how the key concepts are connected, and selecting mathematical concepts/variables. The next phase results in manipulation of the *mathematical representation* into a *mathematical solution*. Finally the solution has to be translated back into the terms of the original *problem*.

Method

Almost 100 students in the first year of upper secondary school at three different schools from a mid sized municipality in the middle of Sweden participated in the study. The three classes attended either the social science or the natural science program.

The main focus, which is to investigate students' communication and interaction regarding the mathematical content, places the study in an interpretative paradigm (Ernest, 1994). To grasp the students' interpretations of the communication, they were gathered in small groups (three to four in each group) and they had to solve a number of tasks. The tasks were chosen depending on what was discussed during the classroom observations, which preceded the group sessions. To create a situation where the students had to discuss and communicate the mathematical content, as well as to challenge them as a group to solve tasks they might have been unable to solve individually, the tasks had to have a somewhat greater level of difficulty than the students were accustomed to. To make it possible to study their interpretations, the external expressions of these interpretations will have to serve as material. That is to say, it is what students communicate, in words, actions, writing and gestures that make up the researchable data.

The problem solving situations with the student groups were video and audio recorded and some field notes were collected. Students' solutions from the algebraic problem solving were also collected to serve as a basis for the analysis.

The video and audio material was scrutinized several times and most of the material was transcribed verbatim for further analysis. In addition to oral communication, relevant non-verbal actions and interactions were included in the transcripts. The transcripts were scrutinized and categorized. In the first step the problem solving situations were divided into three different categories, one for each of the three transitions between the three boxes in the modelling cycle (see figure 1).

1 Problem → mathematical representation

2 Mathematical representation → solution

3 Solution → context and problem

Then each category was analysed regarding the students' discussion about the mathematical content. The results will be presented with examples from each category.

Student task

The students were given a number of algebra problem solving tasks that have been used in national tests in Sweden. All the tasks were algebraic problem solving tasks and the particular task presented below was chosen for this paper because it was a little more difficult than the problem solving tasks in the text-book and it includes all the steps in the modelling cycle. The students have to interpret the context to understand the formula. After they have solved the task they have to interpret their answer and explain the formula using their own words.

> When a freezer is turned off, the temperature inside rises. The following formula can be used to calculate the temperature (*y*) in degrees Celsius after the freezer has been turned off for *x* hours.
>
> $y = 0.2x - 18$
>
> a Find the temperature inside the freezer if it has been turned off for two hours.
> b How long has the freezer been turned off if the temperature inside it is 0° C?
> c Explain in your own words what the formula means.

(Skolverket, 2005, p. 4)

The algebraic formula $y = 0.2x - 18$ includes two variables, one independent (x) and one dependent (y). The formula represents a function because to every value of x there is exactly one value of y. In part a) what is needed it to replace x with 2 and calculate which temperature that corresponds to. In part b) one possible solution would be to construct an equation, $0 = 0.2x - 18$, and solve this for x. Another possibility would be to invert the original function and express x as a function of y. Then one could substitute the number 0 for y and get the corresponding value of x. The number 0.2 has the unit degrees/hour and that means that the temperature rises 0.2 degrees every hour. It is not explicitly said that the temperature is minus 18 degrees when the time is zero, so this is left to the students to interpret. Also the fact that 18 is subtracted could cause confusion because this must be interpreted as $y = 0.2x + (-18)$. Then the formula makes sense as expressing the final temperature as the sum of the increase in temperature after x hours and the initial temperature.

Students' solutions of the task

The results and analysis are based on the categories in the modelling cycle and will be presented with three short episodes from two different student groups' discussion of the task. The excerpts are selected based on their content and how the students treated the mathematical content. These are examples to show how the students reason.

Transition between problem and mathematical representation

The first and the third episode are taken from the same group, consisting of Lollo, Johan, Chris and Per. In the first episode they are in the process of translating the context to the mathematical representation.

Episode 1

The group has started with task a, discussing the meaning of x and y.

1:1 Lollo: y is degrees
1:2 Johan: What isn't y hours?
1:3 Chris: Thus y is degrees and x is hours
1:4 Johan: Yes ...
1:5 Lollo: The freezer has been turned off for x hours
1:6 Per: That is like one fifth
1:7 Johan: Yes one fifth, that is five, six minutes ... No twelve minutes.
1:8 Per: Yes that´s right ... now I thought totally wrong.
1:9 Johan: Oh twelve minutes minus eighteen it is um ... minus ...

Lollo and Chris seem to understand the meaning of x and y [1:1, 1:3] but they do not interrupt Johan when he develops his interpretation of $0.2x$ as 0.2 hours, which he correctly calculates to be 12 minutes. However, it is not certain that Lollo and Chris have another interpretation of x than Johan and Per because Chris says that ”x is hours” [1:3], and that does not necessarily mean ”number of hours”. Johan and Per seem to be convinced about their conclusion [1:7–1:8]. However, Johan gets some trouble when calculating 12 minutes minus 18 [1:9]. Here we may assume that he is not able to find a good interpretation of what his solution means in terms of the situation. Johan's interpretation of x as the unit ”hours” ($0.2x \leftrightarrow 0.2$ hours) instead of $0.2x \leftrightarrow 0.2°/h$ times the number of hours is not an unusual interpretation. This fits within the category ”letter used as an object”, as described already by Küchemann (1981, p. 104). The task here is about making a conversion from $0.2x$, in the discursive and monofunctional register, into ”0.2 degrees per hour times the number of hours”, in the discursive and multifunctional register. This is a non-congruent conversion, since the multiplication sign in $0.2x$ is invisible. However, Johan makes a congruent conversion when translating $0.2x$ ”sign by sign” into ”0.2 hours”.

Transition between mathematical representation and solution

The most frequent solution procedure was the one shown at the beginning of episode 2. The students did not construct an ordinary equation to solve part b. Instead as a part of the solution, they used repeated addition and mental calculation

Episode 2

In this episode Karin, Fia and Sebbe are trying to solve the task.

2:1 Karin: How much did it go down every hour, was it zero point two? Couldn't you calculate so as to get one degree, how many hours that takes? Thus one whole degree and then take that eighteen times?

2:2 Fia: Um

2:3 Karin: Do you understand?

2:4 Sebbe: Yeah, wait, I have to think

2:5 Karin: The question is what one degree is ...

2:6 Fia: Five times zero point two, it becomes warmer. Thus zero point two times five ...

2:7 Sebbe: [Sebbe computes, using the calculator] oh, oh, oh

2:8 Fia: Yeah, but it feels like zero point two. How many, how many hours did it take for it to rise one degree

2:9 Sebbe: Don't you take ...

2:10 Karin: It is five, zero point two, zero point two, zero point two, zero point two and zero point two times ...

2:11 Fia: Yes, I think so

2:12 Karin: So it is

2:13 Sebbe: Don't we take eighteen divided by zero point two ... Yes it is.
[Sebbe takes the calculator and looks at the display]

2:14 Karin: Okay try that

2:15 Sebbe: It is ninety hours

The group reached the solution through discussion when they sort out the problem and Karin starts with the question ”How much did it go down every hour, was it zero point two?” The temperature rises with 0.2 degrees per hour but most of the participating groups come to the conclusion that since the temperature approaches zero it goes down. Their interpretation that the temperature goes down could also have to do with the fact that 0.2 is less than 1, and that a connection is made to the fact that multiplication with a number less than 1 makes the result smaller. The formula, $y = 0.2x - 18$, is given in the monofunctional register but the interpretation is expressed in the multifunctional register (natural language). Karin [2:1] claims that the temperature goes down. Still, her procedure, to find out how many hours it takes to get one degree and then multiply that by 18, would give the correct answer. It is not clear that all students have the same interpretation of the situation. For example Fia [2.8] says that they should find ”how many hours it takes for it to rise with one degree”. However at the end Sebbe come up with another strategy [2:13–2:15] when he divides 18 with 0.2. In this discussion the students move back and forth between registers and although they find the correct solution it is not obvious that they have interpreted the formula correctly.

Explanation of the formula

The third episode involves the same group as episode 1. They have through a long discussion solved parts a) and b). In this episode they are trying to explain the formula (part c) of the task.

Episode 3

3:1	Per:	Yes it is minus eighteen and zero point two and that is like degrees [points at the formula]
3:2	Johan:	y is degrees
3:3	Per:	zero point two x, is the number of hours
3:4	Chris	zero point two is the number of degrees it increases every hour
3:5	Per:	Or decreases
3:6	Chris:	Yes, no you can see that it increases there [points at $0{,}2x$ in the formula]
3:7	Per:	Yes
3:8	Johan:	So then, like ... So, that thing, which is before the x, zero point two is then ... the number ...? [writes at the same time as he talks]
3:9	Chris:	Number of degrees since it is x every hour
3:10	Per:	Yes but what is eighteen?
3:11	Johan:	What, what is eighteen?
3:12	Chris:	That is how much it was ... at the beginning
3:13	Per:	Yes ... eighteen degrees
3:14	Chris:	Minus eighteen degrees

It seems like Per still thinks that $0.2x$ is the number of hours [3:3] and he suggests that the temperature drops [3:5]. This interpretation of 0.2 is common in the data for this study. Many other groups did the same interpretation. Like in the second episode also this group seems to do the same interpretation. It is reasonable to interpret that this connects to the fact that if you multiply something with a positive number less than one, the result gets smaller, than the number you started with. However Chris points at the number 0.2 and explains that it is 0.2 that determines if it increases [3:6]. This seems to be a clarification for Johan too [3:8], that the number, 0.2 is not the number of hours, which Johan stated in the first episode and maintained throughout the whole solving process. Their dialogue in the last lines [3:10–3:14] shows that at least Per and Johan have not interpreted -18 as degrees. They may see it as just a subtraction, which was a common interpretation in other groups. They do not see it, as it is the temperature at the beginning.

Through the solving process and discussion the group is convinced. With help of Chris the group seems at the end to have interpreted the formula.

This is a conversion since the mathematical formula is in the monofunctional register and the explanation in natural language is in the multifunctional register. When the students interpret -18 as subtract 18 it is an example of a congruent conversion. They translate the formula word by word. They should rather interpret it as ”add negative 18” in the formula.

Discussion

In this study the transition between different phases in the mathematical modelling cycle is analysed and the students' obstacles in each transition are characterized. To analyse the data Duval's framework of different semiotic registers is used.

The transition between problem and mathematical representation created some obstacles for the students. They have difficulties to interpret x as ”the number of hours”. Instead they interpret it as just the unit ”hours”. This is consistent with the phenomenon ”letter as object” (Küchemann, 1981). However, seeing it as a conversion from the monofunctional to the multifunctional register, where the invisible multiplication sign in $0.2x$ makes the conversion non-congruent (Duval, 2006), may come closer to an explanation of why the phenomenon ”letter as object” occurs.

In the transition between mathematical representation and solution the students mostly did not create an equation, they rather solved the problem with mental calculation and interpreted the formula term by term. They interpreted 0.2 as the temperature going down, perhaps because 0.2 is less than 1 or perhaps because the temperature approaches zero, which is taken to mean that it drops. However, most of the groups reached a solution after a while, without necessarily having interpreted the formula correctly. The obstacles in this transition occur also in the conversion, when they went between the two registers back and forth, when they should interpret that the temperature rises in the formula, $y = 0.2x - 18$.

The transition from the solution and back to the problem also causes obstacles for the students but is not shown in any episode because all interpretations of the solution were done during the solving process. Many of the groups have difficulties with the plausibility of the answer. They thought that it took too many hours for the temperature to rise.

The difficulties the students faced with explaining the formula were that they interpreted 0.2 as the temperature decreasing. Another obstacle was the meaning of -18. Many students seemed to see it just as a number to subtract. The obstacles occur even in this transition when the students have to change register, from the formula in the monofunctional to natural language in the multifunctional.

To summarize students' obstacles in their problem solving one can conclude that all obstacles that are shown in this study arise when students are forced to go between the registers, and most of the obstacles arise both in transition between problems and mathematical representation, and also between the solution and the interpretation of results.

The phenomena observed in this study have been observed before, but using the framework of Duval made it possible to shed new light on the phenomena, and this could be taken into account when helping students to overcome the obstacles.

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Ways of constructing competence – the cases of ”mathematics” and ”building and construction”

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In this paper I investigate different learning situations in the vocational programme ”building and construction” in upper secondary school in Norway. The aim is to illustrate that what it means to be competent is constructed differently for the same students in (1) the mathematics subject and (2) the building and construction subjects within this vocational programme. Differences are conceptualized in terms of agency, accountability and authority.

The high dropout rate from vocational programmes in upper secondary school is a part of current public debate in Norway. Grade statistics indicate that mathematics is the hardest common core subject for students in vocational education (Utdanningsdirektoratet, 2015), and the subject can thus be determining for the students' completion of vocational education. There is currently a national emphasis on making common core subjects *more relevant* for vocational students in order to decrease dropout (Utdanningsdirektoratet, 2014).

Several international studies have focused on mathematics in school and at the workplace, showing differences in the use of mathematics in the two environments (e.g. Williams, Wake & Boreham, 2001). In Norway some studies have focused on how students in vocational upper secondary education work on workplace related tasks in mathematics (e.g. Sundtjønn, 2013). However, I have not so far been able to find studies that have inquired into what it really *means* to make mathematics more relevant for students in vocational upper secondary education in Norway.

The data and analysis in this paper draw on a case study directed towards the educational programme ”building and construction” in the first year of upper secondary school in Norway. In this case study, I explore learning situations in both (1) the mathematics subject and (2) the programme subjects (meaning subjects directly related to building and construction, for instance the subject ”production” at the school's workshop). I refer to these two environments as different learning communities. Early in the investigation I noticed that the

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way competence was constructed seemed to be different in the two learning communities. The analysis of learning situations in the two learning communities showed that differences in what it means to be competent are seen in: (1) How new knowledge or new ideas enter the community, (2) how students work on tasks in order to be considered competent and (3) how competence is measured in the end products. In this paper I will focus on the part of the construction of being competent that has to do with how new knowledge or ideas enter the community. I will discuss the following question: In learning situations in ”mathematics” and in ”programme subjects”, what characterizes the student role in the process of constructing competence?

Theoretical framework

My framework for analysing competence draws on Gresalfi, Martin, Hand and Greeno (2008). This framework differs in substantial ways from e.g. Kilpatrick, Swafford and Findell's (2001) *strands of mathematical proficiency*. These strands are associated with desirable cognitive changes *in* children so that they can be successful in mathematics. The framework presented by Gresalfi et al. (2008) takes a more collective view. Here competent participation can be seen in "what students need to know or do in order to be considered successful by the teacher and the other students in the classroom" (p. 50). It is important that what counts as *being competent* gets constructed in each classroom, and can be very different from one classroom to another. Thus competence is not only seen as something the student ”has”. Gresalfi et al. (2008) imagine that a *system of competence* exists in classroom activities, and that this system is constructed by negotiation of three aspects:

> [W]e refer to a *system of competence* that is constructed by participants in their practice. This system of competence gets constructed as students and teacher negotiate (1) what kind of mathematical agency that the task and the participation structure afford, (2) what the students are supposed to be accountable *for* doing, and (3) whom they need to be accountable *to* in order to participate successfully in the classroom activity system.
>
> (Gresalfi et al., 2008, p. 52)

Gresalfi et al. (2008) use the term *agency* to refer to how students act, and how they are getting opportunities to act, in the classroom. They draw on Pickering (1995), who proposes different kinds of agency. *Human agency* is associated with attributes such as choice and discretion. *Disciplinary agency* is associated with human passivity, and is characterized by series of manipulations. For instance a person is exercising disciplinary agency in the phase of using well-established procedures (from the discipline) and following predetermined steps. Boaler (2002) uses Pickering's (1995) disciplinary agency to characterize

”traditional classrooms”, where students follow standard procedures of the discipline. She associates ”reform classroom” with students exercising an interplay (or *dance of agency* (Pickering, 1995)) between human agency and disciplinary agency. Here students use their own ideas and methods, and they are positioned to critique others' ideas in addition to using more well established methods from the discipline (Boaler, 2002).

The concepts *accountable for* and *accountable to* are the other dimensions that Gresalfi et al. (2008) use to analyse systems of competence. I will make use of their concept *accountable for* in this paper. This concept refers to what students are accountable for knowing or doing in the community.

Another relevant concept is *authorship*. Who are the authors of ideas and knowledge in a community? Povey, Burton, Angier and Boylan (1999) separate between two epistemological perspectives important when analysing pedagogical practices in classrooms, *external authority* and *author/ity*. They claim that viewing classroom experiences through this lens will help us understand how students experience pedagogical practices. *External authority* is associated with experiences of the authority as external to the students and belonging to experts.

> Meaning is taken for given and knowledge is assumed to be fixed and absolute rather than contextual and changeable. The knower is deeply dependent on others, especially authoritative others.
>
> (Povey et al., 1999, pp. 233–234)

The contrasting perspective is that of *author/ity*. This perspective is associated with students experiencing themselves as “members of a knowledge-making community” (Povey et al., 1999, p. 234). The power here is more distributed, and the students' voice is given primacy. Knowledge is not seen as given, but constructed in the community. External sources are consulted, but students are responsible for being critical of these ideas. The students in the community are themselves *authors* of ideas and knowledge. Knowledge in this perspective is contingent and contextual (Povey et al., 1999).

Methodology

The research is situated in a constructionist paradigm where social phenomena and meaning are continually constructed by the social actors involved (Bryman, 2008). The design used is multiple case study (Stake, 2006), where intrinsic interest in exploring present cases is of importance. The study is of an exploratory character, and the aim is to represent the cases explored, not the world (Stake, 1994). Students in one class enrolled in the educational programme ”building and construction” are the subjects of the study. The units of analysis are these students' *participation and their experiences related to participation* *in the mathematics classroom, and in the programme subjects*. The two contexts are considered as different cases, even though the same students are present in both. There are different teachers in the two communities.

The school was chosen because it was a large upper secondary school having both university-preparatory and vocational programmes. It was of importance that its location made it possible for me to spend substantial time on site, offering more opportunities to learn (Stake, 1994). The school's willingness to participate was also crucial. The class in the study consisted of 14 boys at the age of 15 to 16. I have no indication that this class was outstanding in any way. The class was followed for one year, in both the programme subjects and in mathematics. Data were collected mainly by use of participant observation and semi-structured interviews. I also collected students' work. Mathematics lessons and interviews were audio taped. Audio recording was however not practically possible in the workshop where there was a lot of noise and students were constantly moving. Data from lessons in the programme subjects were therefore gathered by taking notes and pictures during observations, and by audiotaping discussions and interviews with students and teacher during or directly after the lessons.

In the analysis I have followed Postholm (2010) in separating between a *descriptive analysis* where the data are structured, and a *theoretical analysis* where substantive theory is used to analyse parts of the material. In the descriptive analysis, each case was analysed individually by use of constant comparison (Corbin & Strauss, 2008). The aim was to explore what it meant to be competent in the two different communities. This part of the analysis was highly inductive, trying to set aside preconceptions. One core category that emerged from each community concerned how new knowledge entered the community. This was "Use the teacher's method" from the mathematics classroom, and "Giving ideas and being critical" from the programme subjects. In the following I will use the theoretical framework described in the previous section to interpret some of the data from these core categories. This is then a part of the theoretical analysis.

Constructing competence: different learning situations

Use the teacher's methods: an example

The examples in this section are taken from audio taped observations in the mathematics classroom. The mathematical theme for the lessons chosen for illustration is percentages. The teacher's aim for the lessons is presented to me in an interview the day before class as follows.

> When I calculate with percentages, I often think that you have the percentage, the part, and the whole. [...] [The aim is] to teach them three methods

> for finding how much it is if you have a certain percentage of something, and to teach them to find the percentage ... and to teach them to find the whole if you have the part and the percentage. [...] The aim is that after the two lessons tomorrow they are able to separate between those three ways to calculate.

When the teacher formulates that the aim is "to teach them three methods" this says something about *authority*. It lies with the teacher, and is thus *external* to the students. Meaning is also fixed, and the three fixed methods are what the students are *accountable for* knowing.

The first method is introduced to the students in the following way:

Teacher: We have three operations related to percentages that we have to learn. One of the methods, one of the things we have to figure out [...] is to find how much for example 20 % is out of 200 kr [Norwegian kroner, NOK]. [...]

Students: 40

Teacher: Yes, some sees this quite easily, and some needs a way to do it. What I think is best, is to find the percentage factor.

Here it is evident that a few students have solved the mathematical task, but the teacher shows no interest in how the students thought to come to this solution. The teacher claims the *authorship* for the mathematical idea by showing the class what he thinks is the best way to proceed. The students' voices are silenced. *Authority* is *external* to the students. The teacher continues (following directly from the excerpt above):

Teacher: What is the percentage factor in this case? How can I find the percentage factor?

Student A: Divide or multiply?

Teacher: Yes, how to get from percentage to percentage factor?

Student A: Divide ... or ... Multiply!

Teacher: No, what have I done here? [Points to an example for calculating percentage factor on the blackboard.]

Student A: Then I divide.

Teacher: Yes! Then ...

Student A: You reacted so I switched!

In this section the teacher is guiding the students through his own method for finding how much a certain percentage of an amount is. When the students can't answer the question about how to find the percentage factor, the teacher points to an earlier example and a procedure authored by the teacher himself on the blackboard. Again authority is external to the students. The *agency* can be termed *disciplinary*. Instead of giving attention to the meaning of the concept "percentage factor", the teacher forces the attention back to the predetermined steps on how to perform the calculation. The last utterance from the student

“You reacted so I switched” is reinforcing the epistemological perspective of *external authority*. The student is not sure how to find the percentage factor (or what percentage factor really is) and is using the teacher’s reaction to decide whether the right operation to use is dividing or multiplying. This indicates a view that knowledge is fixed, and that the teacher holds this knowledge.

The teacher goes on presenting the steps in his method:

Teacher: Yes, that’s fine. But it’s only you [still in conversation with Student A] who are responding. I would like others to participate as well. [...] Divided by 100, right? Make the percentage factor. So the percentage factor is 20/100 (writes on the blackboard) and that is 0.20. [...] And then we are going to use this further on, because we have the percentage factor. And if we are going to find how much 20 % is out of 200 kr [...]. We find it by taking the 200 kr and multiply [...] multiply by the percentage factor. [...] Now, someone has given the answer several times. What is it?

Students: 40

Teacher: 40 kr, yes. [...] One of the methods we should know is to find how much a certain percentage is out of something. If you are going to find how much 70 % is out of the students in this class, you should be able to find that.

Student B: Then we divide by 100 and multiply by 70.

Teacher: Yes, take the amount – you have to know it. Then it is just to take the amount you have and multiply it by the percentage factor. Then you find how much it is.

As the teacher says, the answer to the teacher’s original task has been given several times. But the main purpose has not been to find a solution. It has been to present the *teacher’s method*. In the last excerpt the teacher is formulating a similar task as the one given before, but one student (Student B) shows another way of thinking. The teacher acknowledges this way of thinking weakly by saying “Yes, [...]”, but quickly draws the attention back to his own solution path. Again he claims authority, and reinforces the epistemological view of *external authority* in this learning situation. The student who presents an alternative way of thinking is silenced.

In this lesson the teacher first presents his three “methods” in the same manner as shown above. For the rest of the lesson, the students are occupied with solving tasks related to use of the three methods. This approach is typical for the lessons observed in mathematics.

Giving ideas and being critical: an example

An agro-technical exhibition was arranged in the school’s neighbourhood and the class was asked to construct some equipment for this exhibition. The order was to construct a stage, rail fences and signboards. The activity was quite complex. I have constructed narratives of how students worked, building on observations in the workshop and interviews with both students and the building and construction teacher.

The order from the customer is presented to the students, and the teacher engages the whole class in making strategies for how to build the stage. The order is to make it 6 m long and 3 m wide. The teacher suggests that they make the stage using Euro-pallets that they put close together in layers. Then they will cover the Euro-pallets with plates on the top and on the sides. It is emphasized by the teacher that it is important to cut as few plates as possible for economic reasons (make it possible to reuse plates). How many Euro-pallets do they need to order? How are they going to put the Euro-pallets together to make the stage as stable as possible? How are they going to join together plates on top to make the stage stable, and at the same time cut as few plates as possible? The whole class is divided in groups, and they are asked to come up with possible strategies.

When groups are asked to come up with solutions they are given *authority* in the form of *author/ity*. Engaged in teamwork, students discuss ideas or strategies. Here more students get a voice, compared to full class situations.

The different groups' solutions are presented to the class. Every idea is acknowledged and considered. All students are responsible for evaluating the different solutions, being attentive to economics and stability.

In terms of agency in general, this phase of the work can be characterized by *human agency*. The students are engaged in processes of presenting solutions, being critical and making decisions. The activity requires more from the students than following a procedure. They are trusted to contribute with their professional knowledge, strategies and critique. Here it is not the teacher who holds the (one) solution. Authority for making solutions is distributed to all the students, and the activity is guided by students' voices and *students' author/ity*. In this activity students are *accountable for* providing professional knowledge, making suggestions for possible solutions, being critical and participating in decision making about which plan to follow.

When the class had decided how to make the stage, agency is shifted more towards *disciplinary agency*. The plan is made, and the students have to get hold of Euro-pallets, put them together, cut plates and assemble the stage.

Work continues on the stage, but more planning and work is needed elsewhere. The teacher teams up four students to make the fence components. One of the students in this team explains:

> We were provided with one of those rail fences that we were supposed to make more of. So it was eight that we had to make. So we had to find a way to make them, the easiest way possible.

The students are given the template, but as they say, they have to "find a way to make them". The students are given time and responsibility for making the fences, including finding out how to do so. In this activity the students are given *author/ity* to author their own solutions to a given task.

Figure 1. *The rail fence component template given to students (in front), and fence components in progress made by students (behind)*

One practical problem was that the students did not have a milling machine to make the trace seen in the bottom board in figure 1. One of the students explains how they used available tools and materials to find an alternative solution:

Student C: [...] So we came up with a solution to bind it together in another way.

Student D: We used 2” x 8” [Student H: For the bottom.] and then we cut ... So we put one of the poles down, and then we used a compass saw to cut around it. [...]

Christina: Who was it that ... Did you figure this out all by yourselves?

Student C: The teacher was there, of course, but ...

Student D: We came up with this. That we could use 2” x 8” and… Was it 3”?

Student C: 2” x 8” here and there is 2” x 3” on the top, which we made the cuts in. In this way we are building upwards on this [refers to the 2” x 8”]. [...]

Student D: Yes, so this one [refers to the fence rail made by the students] is way more stable than the one we got [refers to the template].

The students here claim *author/ity* for finding the solutions in saying “We came up with this”. The knowledge is locally constructed by the students, under local conditions. It follows that the activity can be characterized by *students’ agency*. At the time students were given the task, there were no predetermined steps to follow. They had to find the best way to deal with the practical challenges. At the same time they had in mind professional standards (stability) when doing this work. The students were *accountable for* finding solutions or make a plan, and to make a good product. They had to make use of their own knowledge about tools, materials and constructions to so. The last utterance from student D, shows pride and ownership to the solution.

After solving the practical issues and after having made the first rail fence component, the activity shifts character to more *disciplinary agency*. The students claim that they used four hours to make the first component, and used less than a day to finish the last seven.

Not all activities in the programme subjects have the same authentic degree as the ones described and the students are not always given this much responsibility. Students have to learn to use tools, for example, and are involved in activities where the teacher is more instructive. However, I have tried to illustrate that in the workshop I see students being involved in activities were they are positioned to exercise a considerable degree of freedom and discretion.

Conclusion

The tasks and the ways that the teacher is leading the communication in the mathematics classroom leave little room for human agency such as opportunities for students making choices or showing or sharing insights. *Agency is disciplinary* and for the students "associated with human passivity and characterized by series of manipulations" (Pickering, 1995, p. 115). I have argued that students are *accountable for* knowing the teacher's method, and that the *authority is external* to the students in this learning situation.

In the episodes from the building and construction workshop I will characterize agency as a *dance of agency*. As described above both elements of *human agency* and *disciplinary agency* are in play. Responsibility for authoring solutions is to a large extent given to students, so the community have a high degree of students' *author/ity*. The students were *accountable for* making plans and critically finding solutions according to both professional standards and local conditions.

During the whole study it became clear that making the mathematics subject more relevant to students, can be more than giving content a vocational flavour. It can also be about changing the classroom interaction to support identities of autonomous future workers, giving room for students' agency and author/ity. This can be important for two reasons: (1) In mathematics the students' observed identities can be associated with roles as "received knowers" and a lack of agency (Boaler, 2002). This can be problematic and alienating especially when students' experiences with what it means to be competent in building and construction (their preferred profession) is connected to their ability to exercise agency and *author/ity*. (2) Boaler (1998) inquired into two schools where one used a traditional, textbook approach and one used open-ended activities in mathematics. She showed that in the school with the open approach, students needed to think for themselves, interpret situations, choose, combine and adapt different procedures. These characteristics give me associations to *human agency, author/ity* and the "programme subjects". Boaler (1998) concludes that this kind of working and thinking gives students an advantage when they need to solve problems in new settings. This is highly relevant for future workers.

A consideration of how mathematics in vocational education can be more supportive for the students' professional identities as autonomous workers and better prepare the students for the demands of problem situations in the real world could be an interesting avenue for future research.

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Exploring the role of representations when young children solve a combinatorial task

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This paper is about the representations young children spontaneously use when they are solving a combinatorial task. The paper describes connections between the representations used by the children and how they solve the combinatorial task, and considers whether the results from studies regarding representations of quantity also apply to combinatorial tasks. Our results indicate some connections between the representations used and the solutions presented, but these connections do not seem to apply to the results from studies of quantity. Some possible explanations for this are outlined in the paper, but more studies will be needed to further elaborate on these issues.

The focus of this paper is on the representations young children spontaneously use when they are solving a (for them) challenging combinatorial task. Most research on young children and mathematics has focused on numbers and quantitative thinking (Sarama & Clements, 2009) and studies on children's representations are often connected to quantity. Thus, there are many studies on young children's representations within the context of quantity but few studies on young children's use of representations when solving tasks within other mathematical areas.

Studies of young children's representations often focus on informal and formal representations. One line of inquiry has looked into linkages and/or development opportunities between informal and formal representations (for example Hughes, 1986; Heddens, 1986; Carruthers & Worthington, 2006). Another line of inquiry has focused on connections between representations used by children and their mathematical abilities (for example Piaget & Inhelder, 1969; Carruthers & Worthington, 2006). In this paper both lines of inquiry will be addressed and the following questions will be elaborated upon:

– Do results from studies of young children's representations of quantity also apply when young children solve combinatorial tasks?

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– Are there any connections between the representations young children spontaneously use and how they solve a combinatorial task?

Representations

In the paper we deal with the graphic representations young children spontaneously use when solving a combinatorial task. We will not deal with representations of other modalities such as sound, manipulatives or gestures. In the remainder of the paper, then, the word ”representation” will refer only to graphic representations.

Different researchers have divided and identified representations used by children in various ways, and in this section we will present some of these. Then, in the next section, we will connect these to the combinatorial task that will be described in the paper.

> A representation is typically a sign or a configuration of signs, characters or objects. The important thing is that it can stand for (symbolize, depict, encode, or represent) something other than itself.
>
> (Goldin & Shteingold, 2001, p. 3).

Children's drawings are a first step towards using representations since they refer to objects, events, ideas and relationships beyond the surface of the drawing (Piaget & Inhelder, 1969; Matthews, 2006). Children do not distinguish between marks used for writing, mathematics or drawing; they often combine them, and their trials, inventions and combinations are important in developing their understanding of abstract symbolism in mathematics (Carruthers & Worthington, 2006).

Piaget & Inhelder (1969) distinguished between two types of representations used by children: symbols and signs. Symbols include pictures and tally marks that have some resemblance to the objects referred to. Each child can invent such symbols since they are not conventions of society. Signs are conventions of society in the form of spoken and written symbols that do not resemble the objects represented.

As mentioned, studies of how children use representations have often been connected to quantity. One influential study regarding representation of quantity was conducted by Hughes (1986), who investigated how children use their own marks when representing numerals. He identified four forms of marks used by children to represent quantity: idiosyncratic, pictographic, iconic and symbolic. Idiosyncratic marks are irregular representations which cannot be related to the number of objects represented. Pictographic representations are pictures of the represented items, while iconic representations are based on one mark for each item. Symbolic representations are standard forms of representation, for example, numerals and equal signs. The same types of children's own representations have been identified in later studies, for example by Carruthers and Worthington (2006). Carruthers and Worthington also identified dynamic and written representations.

Representations often have some kind of relation to objects. Heddens (1986) focused on the connection between concrete (objects) and abstract (signs) representations. He introduced pictures and tally marks as two levels between concrete and abstract representations. He referred to representations of real situations, for example, pictures of real items, as semi-concrete, whereas he referred to symbolic representations of concrete items, where the symbols or pictures do not look like the objects they represent, as semi-abstract.

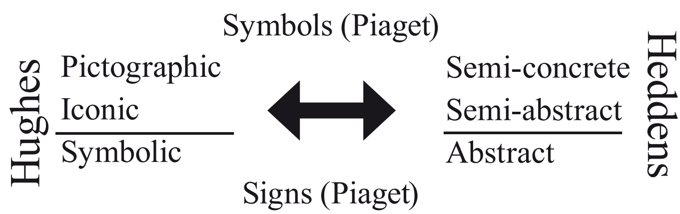

Figure 1. *Connecting Piaget, Hughes and Heddens*

Figure 1 above connects Piaget, Hughes and Heddens. What Hughes named pictographic representations are what Heddens would refer to as semi-concrete, while what Hughes called iconic representations are semi-abstract in Heddens's terms. All of these representations are symbols in Piaget's terms, as each child can invent them. Further, what Hughes named symbolic representations Heddens calls abstract representations and Piaget calls signs. In the remainder of this paper, Hughes's terms will mainly be used when presenting and analysing the results.

Combinatorics

Combinatorics concerns determination of a finite number of discrete structures. These structures offer the opportunity to explore simple combinatorial problems with young children. Combinatorial tasks can also serve as a base on which to build understanding within other areas of mathematics, such as computation, counting and probability. Furthermore, combinatorial tasks facilitate systematic thinking as well as making conjectures and generalizations (English, 2005). The combinatorial task in this study is of the type enumerative combinatorics: counting permutations, in this case for $n = 3$.

Research on young children and combinatorics started in the late 1950s with Piaget and Inhelder, who investigated the cognitive development of children's combinatorial and probabilistic thinking. They concluded that children (age 7–8) were not able to work with 2 x 2 or 3 x 3 permutation problems. Later research, however, showed that within a proper and meaningful context, pupils indeed could work effectively with combinatorial situations finding permutations (English, 1991, 2005). The task presented in this paper is embedded in a problem-solving context, which already in the late 1960s was seen as suitable to combine with the practical element within combinatorics (English, 1991).

The major difficulty for young children when solving combinatorial tasks is in listing items systematically (English, 2005). Based on empirical investigations, English (1991) identified five strategies used by young children when working with combinatorial tasks: 1) random selection of objects – with duplicates, 2) trial and error with random item selection – with rejection of duplicates, 3) emerging pattern for the choice of objects – with rejection of duplicates, 4) consistent and complete cyclical item selection – with rejection of duplicates and 5) ”odometer pattern” in item selection. Some of these hierarchical strategies where children start to emerge pattern for the choice of objects are more effective than others when it comes to finding all possible combinations.

The study

The results presented in this paper are from a design research study investigating how to teach mathematics through problem solving in preschool classes. Design research is a cyclic process of designing and testing interventions situated within an educational context (Anderson & Shattuck, 2012). The task in focus in this paper was the third task of six that the children worked on during the intervention. They had already worked on two challenging problem-solving tasks, but not with any combinatorial tasks.

Preschool class

The task was conducted in six preschool classes with a total of 87 children. The classes were selected based on the interest of the teachers at the schools. The Swedish preschool class was implemented in 1998 to facilitate a smooth transition between preschool and primary school and to prepare children for further education. There are no regulations or goals around the teaching of mathematics, but the content of both the preschool and primary school curriculums are to form the basis of the preschool’s activities. The working methods and pedagogy are not supposed to be either like school (with a tradition of learning) nor like preschool (with a tradition of play) but a combination of the two (Swedish National Agency for Education, 2014).

The bear task

The combinatorial task on which this paper focuses required children to consider how many different ways three toy bears could be arranged in a row on a sofa. To make the task meaningful for the children it was presented as a conflict between the toy bears, where they could not agree on who should sit at which place on the sofa. One toy bear then suggested that they could change places every day. The task for the children became to find out how many days they could sit in different ways on the sofa.

The children were divided into groups, where approximately 12 children at the time worked on the task. The researchers acted as teachers during the lesson (one researcher per group). When introducing the task, the children were shown three small plastic bears, one red, one yellow and one green. After the introduction the children worked individually. They were given white paper and pencils in different colours but no instructions regarding what or how to do any documentation on the paper. After working alone first for some minutes and then in pairs the children were gathered for a joint discussion based on their documentations. When working in pairs the children compared their documentations to identify similarities and differences. They did not change their documentations. In the joint discussion the different permutations were explored and a joint effort resulted in a display of the combinations with the plastic bears. Finally, the ways the children had documented their solutions were discussed. The purpose of this discussion was to explore the potential of different ways of representing mathematical thinking, to make the children aware of their own and others' use of different marks and to extend the children's repertoire of representations by using peer modelling. Peer modelling implies focusing on children's own marks, discussing ways of representing, meanings and strengths (Carruthers & Worthington, 2006).

Analysing the data

This paper will focus on the influence of the choice of representation, if any, on how the children solved the combinatorial task. The representations referred to are the ones in the children's documentations on paper as described above. These are what Hughes (1986) named pictographic and iconic representations.

When we categorized the documentations as pictographic or iconic representation, we found it necessary to add a category that included both these representation, as some children had used both types. After this, the documentations in each category were categorized once more based on the solution of the task. The number of permutations was a first classification after which the uniqueness also became a component of concern.

Results

As can be seen in table 1, this was a challenging task for the children, and only two of 87 children found six unique permutations when they worked individually with the task. These two children used iconic representation.

Table 1. *Categorization of children's documentation*

Short name	Explanatory statement	Picto-graphic	Picto-graphic & Iconic	Iconic
No new permutations	The child has drawn some toy bears or the combination shown by the teacher and then no further combinations	3		2
Unique permutations A	The child has drawn unique combinations where the total number of combinations is less than six	15 (4)	8 (3)	24 (10)
Unique permutations B	The child has drawn six unique combinations			2
Duplicate permutations	The child has drawn combinations where one or several combinations are duplicated	3		30 (1)
Total		21	8	58

All of the children used some kind of pictographic and/or iconic representation when solving the task; the majority (58 of 87) spontaneously used iconic representation. These iconic representations were of different kinds but always in the colours of the toy bears. Most of these children drew circles or lines, but a few also replaced the toy bears with hearts.

Few children used both pictographic and iconic representations (8 of 87). All of them started with pictographic representation and changed after one or two permutations to some form of iconic representation.

The numbers in parentheses in the category *unique combinations A* represent the number of children who drew three unique combinations and no more, where each toy bear sat at each place once. This indicates some kind of systematization in the solutions as each toy bear is drawn once at each place on the sofa. This was done by 17 children, with four using pictographic representation, three using pictographic and iconic representation, and ten using iconic representation.

None of the children that used pictographic representation had a solution with more than five combinations. This refers both to the children who had unique combinations and those with duplicate combinations. However, the majority of the children that used pictographic representation (15 of 18) produced only unique combinations.

The majority (30 of 33) of the children who made duplicate combinations used iconic representation. Of these, one documentation included the six unique combinations but they were duplicated and thus the child did not seem to recognize the six unique combinations as ”special”.

Discussion and conclusions

In this final section we will focus on the two questions raised in the introduction to the paper:

– Do results from studies of young children's representations of quantity also apply when young children solve combinatorial tasks?

– Are there any connections between the representations young children spontaneously use and how they solve a combinatorial task?

The children in this study had not had any formal instruction regarding representations and, as mentioned, peer modelling was used only after the children had worked on the task individually. When working on the task, all children used pictographic and/or iconic representations. Both of these have a resemblance to the objects they represent (Hughes, 1986). The pictographic representations were drawings of the plastic bears in the three colours, and the iconic representations were made in the three colours.

In studies focused on representations of quantity, pictographic and iconic representations are associated with a lower level of development as they reveal children's attention to each object rather than to the total quantity (Sinclair, Siegrist & Sinclair, 1983). However, in this combinatorial task the children needed to pay attention to each object as well as to the relation between the objects, therefore both pictographic and iconic representations were well suited to it (Hughes, 1986).

The use of iconic representations implies a semi-abstract level (Heddens, 1986); it is more abstract than the use of pictographic representations (semi-concrete level) which were used by fewer children in this study. But, the children who used pictographic representations made fewer duplications than the children who used iconic representations. None of the children who used pictographic representations had a solution with more than five combinations, and the majority of them (15 of 18) drew only unique combinations. Why is that?

Maybe it has to do with time, even though there were no time constraints for this task. It takes longer to draw toy bears than to draw iconic representations, thus the children who used pictographic representations (drew toy bears) had more time to think. Further, drawing toy bears can be experienced as more real, and when working on the task the children who drew the bears could be heard saying things like, ”Now it is your turn to sit in the middle” and ”He has already been in the middle.” Iconic representations are easier to draw, which makes the process faster and maybe that is why the solutions with iconic representations contained a lot of duplications, even when the combinations drawn are few. For example, some children drew three combinations with iconic representations, two of which were duplicates.

Thus the time issue and the connection to real toy bears may be why children who used pictographic representations made few duplications.

According to Devlin (2000), differences in how individuals solve what may look like ”the same” mathematical task can be connected to how the task is described and what it pertains to. As mentioned, the bear task was presented as a conflict between the toy bears, where they could not agree on which of them should sit at which place on the sofa. For some children this seems to have made the task familiar. Saying things like ”Now it is your turn to sit in the middle” and ”He has already been in the middle” indicates an interpretation of the task as a real and familiar situation, and this may be why few duplications were made.

According to English (2005), the major difficulty for young children when solving combinatorial tasks is listing items systematically, and yet another explanation is that the main issue in solving this task is not about the representation used (pictographic and/or iconic representations) but about the systematization of the representations. Regarding systematization Piaget distinguished between empirical abstractions and constructive abstractions (Kamii, Kirkland & Lewis, 2001). Empirical abstractions are generated from empirical experiences where the children focus on certain properties of objects (for example, colour) and ignore others (for example, size). Constructive abstractions are generated from mental actions on objects, not from the objects themselves. Distinguishing relationships between objects when solving a combinatorial task is an example of a constructive abstraction. This means that the relationships that the children need to figure out in the combinatorial task are retrieved from mental actions on the objects. Regardless of whether they use pictographic or iconic representations, the children need to mentally list items systematically to keep track of which combinations they have and have not drawn. Such mental actions on objects are time consuming, which again can be connected to the time issue when drawing toy bears instead of iconic representations.

To sum up, there seem to be some connections between the representations young children spontaneously use and how they solve a combinatorial task. However, these connections seem to differ from the results from studies of young children’s representations of quantity. Iconic representations do not generate a higher level of solution of the combinatorial task; quite the opposite, pictographic representations do seem to imply more systematization and less duplication. Some possible explanations for this have been outlined above, but more studies will be needed to further elaborate on these issues.

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Large-scale professional development and its impact on mathematics instruction: differences between primary and secondary grades

Jannika Lindvall

Worldwide, substantial resources are spent on professional development [PD] for mathematics teachers. Still, the recommendations regarding effective PD are not specific enough to support practice and more research conducted on a larger scale and in multiple contexts is still needed. In this paper we report on a PD-program working together with over 10 000 students, 400 teachers, principals and municipality leaders. The results from surveys indicate that the teachers report having made changes in their mathematics instruction in line with those advocated in the PD. However, statistically significant differences can be seen between teachers from different grade levels. These differences are further discussed in relation to a set of core critical features of effective PD.

Over the past 25 years, the perceptions about what mathematics students should master and how they should learn it have changed. School leaders around the world are under growing pressure to reassure that students' results in mathematics improve (Even & Ball, 2009) and there is a tendency to move away from traditional to more inquiry-based approaches to teaching (Clewell, Cohen, Campbell & Perlman, 2005; Goldsmith, Doerr & Lewis, 2013). Professional development [PD] has been regarded as a key to improve the quality of education and in many countries substantial resources are spent on PD-programs for teachers (Desimone, 2009). In view of these wide investments, there is certainly a need for a strong base of research in order to guide policy and practice.

Within the research literature, there seems to be a consensus regarding some core critical features of effective PD, for example that it should be sustained, coherent with school and state policies and address both the subject specific content as well as how to teach it (e.g. Desimone, 2009; Marrongelle, Sztajn & Smith, 2013; Timperley, Wilson, Barrar & Fung, 2007; Wayne, Yoon, Zhu, Cronen & Garet, 2008). However, despite this general agreement, it is argued that the existing research on PD lacks sufficient specificity to support policy and practice (Cobb & Jackson, 2011; Wayne et al., 2008). Though the existing studies have contributed much to our understanding of effective PD, we still

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need more knowledge regarding PD conducted on a larger scale, with non-volunteers and in multiple contexts (Clewell et al., 2005; Goldsmith et al., 2013; Marrongelle et al., 2013; Wayne et al., 2008). For instance, research on PD for science teachers suggests that it may need to look different depending on the grade band (McNeill & Knight, 2013). Therefore, rather than focusing on the effectiveness of a PD-program as a global characteristic, future research should pay attention to how it works in particular settings and for different teachers (e.g. levels of experience, subject, grade level) (Chval, Abell, Pareja, Musikul & Ritzka, 2008; Goldsmith et al., 2013; Timperley et al., 2007).

In this paper, we examine a large-scale PD-program for mathematics teachers. The results from a previous study (Lindvall, 2016) indicate that this PD has affected the primary and secondary grade students' achievement in different ways. While the primary grade students (grade 1–5) show a small improvement, the students in the secondary grades (grade 6–9) demonstrate a declining trend. In order to gain a deeper understanding of these results, it becomes interesting to study how the program may have affected other issues related to mathematics teaching. Therefore, the aim with this paper is to examine and compare how the PD has affected the participating primary and secondary teachers' reported mathematics classroom instruction. The results are further discussed and used to elaborate on a set of core critical features of PD identified in the literature.

What do we know about effective PD?

Even though scholars have discussed various characteristics of effective PD, recent research seems to point towards a larger agreement on some core critical features (Timperley et al., 2007; Wayne et al., 2008). In fact, Desimone (2009) argues that there exists a research consensus on five main critical features of PD primarily associated with changes in teacher knowledge and practice. Below, these features are described and will later be used to characterize the PD-program and to discuss the results. The reasons for this is because there seems to be a consensus about these features in the research literature and, as argued by both Desimone (2009) and Wayne et al. (2008), using a shared set of features allows for researchers to build on each other and thereby extend our knowledge. The features *content focus* and *coherence* are given more attention since these are the ones that differ primarily between the grade levels. Thereby, they are particularly supportive for discussing the results in relation to the purpose of the study.

The first feature, *content focus*, may be the most influential factor of PD-programs' impact on teacher learning and student achievement (Desimone, 2009; Timperley et al., 2007). Still, we know very little about the actual content taught in various PD-programs and how this may influence teachers' instruction (Scher & O'Reilly, 2009). In the past years, a number of reviews and meta-analysis (e.g. Clewell et al., 2005; Scher & O'Reilly, 2009; Slavin &

Lake, 2008) have suggested that PD which put emphasis on both the subject knowledge and how to teach it are the most effective ones. This type of teacher knowledge, also referred to as pedagogical content knowledge [PCK], concerns the subject content in relation to knowledge of the students in class, the curriculum and teaching (Ball, Thames & Phelps, 2008). However, the concept of PCK is broad and can be divided into several subdomains (Ball et al., 2008). For example McNeill and Knight (2013) have shown that teachers participating in the same PD, but within different grades, experience unique challenges that need to be addressed for their particular contexts. Also, Chval et al. (2008) revealed that teachers themselves are asking for PD that is focused on the content and the grade level they teach.

In addition to content focus, Desimone (2009) argues that the teaching practices advocated in the PD should be *coherent* with teachers' knowledge and beliefs. Similar reasoning is brought up by Penuel, Fishman, Yamaguchi, and Gallagher (2007), who write that teachers are more likely to make changes in their instruction if the teaching practices advocated in the PD are aligned with the teacher's own goals of learning and their goals for students. Another important aspect of coherence is the PDs' consistency with the teaching practices endorsed in school and state policies (Desimone, 2009; Penuel et al., 2007).

Furthermore, effective PD should engage teachers in *active learning*, such as analyzing classroom videos or conducting mathematical lessons. Finally, it should include sufficient *duration* and *collective participation*, where teachers from the same school, grade or department all take part in the in-service education and meet regularly during an extended time-period. (Desimone, 2009)

The PD-program

The project reported on in this study is a combined research and development program in cooperation with a university and a larger municipality in Sweden. The overarching goal is to establish an effective mathematics education within the municipality to ensure that all students receive the best possible conditions to develop mathematical skills and knowledge. The project includes several elements, such as PD for principals and establishment of new routines (e.g. annual formative tests in mathematics) and new positions (e.g. heads of mathematics at every school). In this paper, focus is put on the teacher PD-program. Within a five-year period, all teachers teaching mathematics at a public elementary school should have participated in the PD. Here we report on the results from teachers who participated during 2013/2014 and 2014/2015.

The PD-program's design can be described in relation to Desimone's (2009) five critical features of effective PD. Regarding *content focus*, the program is concentrated on teachers PCK specific to mathematics. Special attention is directed towards teaching for the mathematical competencies set out in the national curriculum (Skolverket, 2011). These five competencies are related to mathematical concepts, methods, reasoning, communication and problem-solving. In order to make progress towards this focus, two main tracks in the program are formative assessment (cf. Wiliam, 2011) and teaching mathematics through problem-solving (cf. Stein, Engle, Smith & Hughes, 2008).

Further, though we cannot determine a *coherence* between the PD and teachers' knowledge and beliefs, several actions have been taken in order to establish a coherence with school and state policies. Firstly, the program focuses on teaching practices in line with the national curriculum (Skolverket, 2011). Secondly, it involves a joint PD for principals and subject representatives. Thirdly, regular meetings devoted to discussions about the projects impact and future are held between politicians, principals, teachers and researchers.

Finally, the program's design stresses *duration*, as well as an *active learning* and *collective participation* among the participating teachers. All teachers teaching mathematics are expected to participate in the PD-program, which takes place locally at every school. During their year of participation, teachers meet for two hours every other week to engage in activities such as analyzing mathematical lessons and setting up annual plans for the mathematics instruction. These discussions are further supported by doctoral students in mathematics didactics, who also works as mathematics mentors within the municipality.

Data collection and analysis

In order to analyze and evaluate the project, multiple sources of data are collected on a regular basis. Here, results obtained from teacher surveys are reported. Almost all mathematics teachers at the respective schools, with the exception of a few percent that were given permission by their principals to abstain, participated in the PD-program. During the first (pre) and last (post) session they were asked to fill in a survey and the response rate was more than 99 %. In this paper, data from the primary (n = 83) and secondary grade teachers (n = 26) who attended both sessions are used when describing the projects impact on their reported instruction. Questions on teachers' perceptions about the PD were not included in the pre-surveys, whereby only data from the teachers attending the last session ($n_{primary} = 104$, $n_{secondary} = 31$) are included when reporting on these results.

The questions in the surveys concerned, among other things, teachers' perceptions on collegial cooperation, curriculum materials and their students. In this paper, emphasis is on questions regarding teachers' reported mathematics instruction occurring in the classrooms, as opposed to e.g. homework. On a four-point scale, they were asked to state how often they conduct certain activities in their mathematics classrooms. We also report on results from questions which only appeared in the post-surveys and that regard teachers' views of the PD and its implementation. The analyses of those questions showed large differences between teachers from the primary and secondary grades, whereby this data provides important information that supports the discussion of the study's results.

In order to assess if the PD-program had any impact on the mathematics instruction, paired sample *t*-tests were conducted for teachers in the primary and secondary grades respectively. Further, to assess the strength of the impact, as well as facilitating comparison between the two groups of teachers, the effect sizes were calculated in terms of Cohen's *d*. For interpreting the results the guidelines proposed by Cohen (1988) were used, with .2 representing small effect, .5 representing moderate effect and .8 representing large effect. Finally, independent sample *t*-tests were employed to compare the primary and secondary grade teachers' perceptions of the PD. Also here Cohen's *d* was used to assess the magnitude of the differences. However, one could argue that the Likert-scales should be seen as ordinal instead of continuous. Hence, also the non-parametric alternatives (Wilcoxon Signed Rank Test and Mann-Whitney U Test) were conducted, but they showed similar results as the *t*-tests.

Results

The aim with this paper is to examine how the PD-program has affected the participating primary and secondary teachers' reported mathematics instruction. The results are summarized in table 1.

For the primary grades, the results show that the teachers, after the PD, report on taking a larger responsibility in leading the mathematics instruction compared to instruction before the PD. For example, statistical significant differences can be seen for statements on how common both lectures and whole-class discussions are during instruction. This is further supported by the fact that these teachers, after the PD, report that their students spend less time on speed-individualized work in their textbooks. However, the effect sizes are all considered to be small. The changes in instruction reported by the secondary grade teachers are in general larger, with the effect sizes varying between large, moderate and small. As the primary grade teachers, these teachers also report on leading more whole-class discussions and students spending less time on individual work in textbooks. However, in contrast to the primary grades, they also appear to have less lectures during instruction. Moreover, they report on students devoting more instructional time to memorizing formulas as well as less time on presenting their solutions to mathematical problems for the whole class.

Further, though not always significant, both groups of teachers report on putting more emphasis on all of the mathematical competencies. However, just as for the questions about instruction, larger changes can be seen for the secondary grades (see table 2). While the primary grade teachers only show small

Table 1. *The primary (P) and secondary (S) grade teachers' reported instruction before (Pre) and after (Post) participation in the PD*

How often do the following activities occur in your mathematics classes?	Grade	Mean (SD) Pre	Post	Change in mean	t (df)	Effect Size
I lead whole-class discussions focused on mathematical concepts	*P*	2.77 (.67)	3.05 (.75)	.28	2.92 (74)	.39**
	S	2.25 (.85)	2.75 (.90)	.50	2.94 (23)	.57**
Students listen and / or take notes when I lecture	*P*	1.44 (.84)	1.90 (1.06)	.46	2.91 (76)	.48**
	S	2.48 (1.05)	2.08 (1.08)	-.40	-2.31 (24)	-.38*
I lecture about new content through formal presentations	*P*	2.42 (.86)	2.68 (.86)	.26	2.43 (73)	.30*
	S	2.72 (.79)	2.60 (.76)	-.12	-.77 (24)	-.15
Students memorize formulas and calculation procedures	*P*	1.82 (.87)	1.97 (.91)	.15	1.26 (73)	.17
	S	2.13 (.76)	2.65 (.94)	.52	2.79 (22)	.61*
I ask the students to motivate and explain how they have arrived at their answers	*P*	3.64 (.56)	3.66 (.53)	.02	.39 (76)	.04
	S	3.35 (.75)	3.58 (.58)	.23	2.00 (25)	.34
I let several students present their solutions to math problems for the whole class	*P*	2.66 (.70)	2.58 (.70)	-.08	-.90 (76)	-.11
	S	2.24 (.78)	1.92 (.64)	-.32	-2.32 (24)	-.45**
I let the students work at their own pace in the textbook	*P*	2.77 (1.20)	2.34 (1.27)	-.43	-3.08 (72)	-.35**
	S	3.20 (1.00)	2.24 (1.23)	-.96	-3.87 (24)	-.85**

Notes. * $p < .05$; ** $p < .01$

significant changes related to competencies concerning mathematical concepts and communication, the teachers in the secondary grades report on moderate changes related to problem-solving and communication.

Finally, the analyses show that the two groups of teachers experienced varying degrees of difficulty in introducing changes in their mathematics instruction based on their experiences from the PD. Firstly, compared to the primary grade teachers ($M = 1.89$, $SD = .94$), the teachers in the secondary grades to a higher extent expressed that they had insufficient opportunities to practice on their new experiences ($M = 2.29$, $SD = 1.04$; $p < .05$, $d = .42$). Secondly, the teachers from the secondary grades ($M = 3.23$, $SD = .81$) perceived the available time for planning as a larger problem than those from the primary grades ($M = 2.77$, $SD = .97$; $p < .05$, $d = .50$). Thirdly, compared to the primary grade teachers

Table 2. *The teachers' reported instructional focus on mathematical competencies before (Pre) and after (Post) participation in the PD*

Mathematical Competency	Grade	Mean (SD) Pre	Mean (SD) Post	Change in mean	t (df)	Effect Size
To use and analyze mathematical concepts and their interrelationships	*P*	3.19 (.63)	3.37 (.59)	.18	2.16 (74)	.30*
	S	3.04 (.75)	3.29 (.69)	.25	1.24 (23)	.35
To formulate and solve problems using mathematics and also assess selected strategies and methods	*P*	3.07 (.91)	3.25 (.55)	.18	1.84 (75)	.31
	S	2.92 (.78)	3.33 (.57)	.41	3.12 (23)	.60**
To use mathematical forms of expression to discuss, reason and give an account of questions, calculations and conclusions	*P*	2.77 (.92)	3.03 (.66)	.26	2.32 (74)	.32*
	S	2.64 (1.00)	3.24 (.78)	.60	2.60 (24)	.67*

Notes. * $p < .05$; ** $p < .01$

($M = 2.15$, $SD = 1.04$), the teachers from the secondary grades ($M_1 = 2.61$, $SD_1 = .99$; $p < .05$, $d = .45$) also reported on the access to relevant classroom resources as a major issue in trying to change their instruction.

Discussion

One aim of the reported project is to move away from the, in Sweden (Bergqvist et al., 2009), dominating traditional approach to teaching (c.f. Boaler, 2002) and thereby give students opportunities to develop *all* the mathematical competencies mentioned in national curriculum (Skolverket, 2011). The findings from this study indicate that this goal has been only partly fulfilled and particularly, that it differs between the grade levels. Compared to the primary grades, the secondary grade teachers report that they have made larger changes in their instruction. But at the same time, it is their students' results that have declined (Lindvall, 2016). The reasons for these differences can be understood from various perspectives. Here the variations are discussed in light of Desimone's (2009) critical features of effective PD. Since the design of the PD is the same for all teachers, the potential differences between the grade levels should be related to its coherence with teachers' knowledge and beliefs as well as school and state policies. However, to only discuss a PD-program's coherence is impossible without connecting it to its actual content. Therefore the critical features *content focus* and *coherence* will be emphasized in the discussion.

To begin with, while Desimone (2009) stresses that effective PD should be coherent with teachers' knowledge and beliefs, the question regarding coherence with teachers' actual practice is not mentioned. For example, the findings that the secondary grade teachers have made larger changes in their instruction may be due to the fact that the teachers in the primary grades started with instructional practices more in line with those advocated in the PD (e.g. more whole-class discussions). In other words, before participation in the project, the content focus of the PD was more familiar to the primary grade teachers. Even if the PD seems to have contributed to the instructional differences between the grade-levels now being smaller, this is not the case for students' results where the primary grades show a small improvement and the secondary grades a decline (Lindvall, 2016).

The coherence between the content of the PD and teachers' initial practice may help explain the rather contradicting results. If considering the results from the pre-surveys, as well as a previous study on mathematics instruction in Sweden (Bergqvist et al., 2009), it is clear that the PD-program places higher demands of instructional changes on the secondary grade teachers. Could it be that these requirements are too high? For instance, even though these teachers report on putting more emphasis on students' problem-solving competencies, the results also indicate that they less often let students present their solutions to mathematical problems for the whole class. These results may seem contradictory. However, as suggested by Schneider and Plasman (2011), specific features of inquiry based instruction are easier to learn whilst others, such as having students pose questions, are more challenging. Thus, the secondary grade teachers may have tried to put more focus on problem-solving, but have not yet reached so far that they use students' solutions as a basis for whole-class discussion. In fact, Stein et al. (2008) write that for teachers who are novices to teaching mathematics through problem-solving, the part including supporting students problem-solving skills during mathematical discussions based on students' solutions to the problem, is a particularly large challenge.

Nevertheless, if one wants to accomplish an instructional change, teachers likely have to be challenged. As seen in Timperley et al. (2007), the PD-programs showing the most positive effects were those who managed to strike a balance between being supportive and being challenging. Since the content focus of the PD in this study seems to be less coherent with the secondary grade teachers' initial practice, and thereby also more challenging, it may be even more important to consider how to support these teachers. Even so, the results indicate that, compared to the primary grades, the secondary grade teachers experience less support from their local context in carrying out changes in their instruction. First of all, the secondary grade teachers to a higher extent express that they experience barriers, such as available time for planning, in carrying out the instructional changes. Secondly, additional analyses of the teacher surveys suggest that the secondary grade teachers have experienced less support from their students, and the students' parents, in trying to make changes in their instruction. Thirdly, these analyses also indicate that the teachers in the primary grades perceive their mathematics curriculum materials as more supportive compared to those in the secondary grades. As argued by several scholars (Cobb & Jackson, 2011; Penuel et al., 2007), instructional improvement at scale is not just about teacher learning, but also regards questions on how schools and broader educational jurisdictions may support or constrain teachers in making the instructional changes. Thus, the results of this study suggest that not only the PD's coherence with school and state *policies* are of importance, but also if it is consistent with and supported by school and state *structures* (cf. Cobb & Jackson, 2012), such as organizational routines and positions.

To finish, even though the secondary grade teachers experienced more barriers in carrying out the teaching practices advocated in the PD, it is still those who report on having made the largest instructional changes. At the same time, it is the results of their students that have declined. Still, if one wants to determine the full effects of a PD-program it is not enough to do a single follow up after one year (Desimone, 2009; Wayne et al., 2008). For instance, Harris and Sass (2011) found that PD attainment in the current year has a negative effect on student results in high-school math, while it becomes positive after two to four years. Future research should therefore look at how teachers, instruction and students in both secondary and primary grades are affected by PD in longer terms.

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What can a re-analysis of PIAAC data tell us about adults and mathematics in work?

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In this study our aim was to investigate possibilities and constraints when analysing data from the international study *Programme for the international assessment of adult competencies* (PIAAC), 2012, with an interest in adults and mathematics in work. Similarly to PISA, the PIAAC study is conducted by OECD, but targeting 16–65 year olds, investigating adult literacy, numeracy and problem solving. We present findings for adults and mathematics in work through patterns that were possible to identify in a quantitative reanalysis of the Swedish background data. We also present an analysis of the background questions per se, drawing on Bernstein's competence and performance models.

In society today there is a great interest in learning, not only formally such as in school, but also in informal learning settings as in workplaces. Moreover, there is a great societal interest in trying to measure mathematics learning and knowing, e.g. through studies like TIMSS and PISA. The political effects of such investigations are criticised in educational research (c.f. Kanes, Morgan & Tsatsaroni, 2014). Simultaneously, outcomes from international comparisons inform discussions of the way forward with regards to, for example, mathematics education (Skolverket, 2014). Time, money and effort are spent in society on such large scale assessments. The most recent international comparison on mathematics and adults where Sweden took part is PIAAC, *Programme for the international assessment of adult competencies,* in 2012 (OECD, 2013) that investigated skills proficiency among adults. These comparisons would be possible to perform without the use of ICT and we address the role of ICT briefly at the end of the paper.

Our interest in this paper is on adults' mathematics in work in relation to PIAAC. Our warrant for this is that there are data sets publicly available for research, such as the answers to the background questions in PIAAC. In these questions, adults are asked about different spheres of life, including working life. The endeavour we undertook in this study was to explore the possibilities

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for learning about mathematics in work from a reanalysis of a selection of the available data, while simultaneously investigating possible constraints in what may be revealed in such analyses. More specifically, our aim was to investigate possibilities and constraints when using data from PIAAC 2012 with an interest in adults and mathematics in work, with these two questions:

1. What patterns concerning adults and mathematics in work are possible to identify through a quantitative reanalysis of Swedish data from the PIAAC 2012 background questions?

2. What characteristics of mathematics in work ”according to PIAAC” are possible to construe through an analysis of the background questions, specifically in relation to the notion of competence?

Mathematics and numeracy

Our interest in this paper is in *mathematics* both as a school discipline and as part of workplace competence (see e.g. FitzSimons, 2002). The term adopted in PIAAC is *numeracy* (OECD, 2013), which is a contested term (e.g. Jablonka, 2015; Sträβer, 2015). In short, we draw on O'Donoghue (2003) who argues for the importance of not reducing mathematics for adults to numeracy in a limited sense, such as basic calculation skills used in out-of-school contexts. He also writes that ”mathematics education should not be defined exclusively in terms of school mathematics. School mathematics cannot be treated in isolation from adult domains such as 'everyday mathematics' and 'workplace mathematics' ” (p. 39).

We have chosen to generally adopt the term ”mathematics in work” with an interest in mathematics taken in a broad sense (FitzSimons, 2002, 2014).

> There is not one single mathematic, absolute and infallible (Davis & Hersh, 1980/1983; Ernest, 1991; Kline, 1980, 1987) but rather a plurality of mathematics which operate on a pragmatic basis, linked to time and place. (FitzSimons, 2002, p. 15).

PIAAC presents and extensively discusses the application of skills at work. For the *numeracy test*, Swedish participants performed above the OECD average, with Sweden as number three of 22 participating countries. What is in focus in this paper, however, is an analysis of the potential application of numeracy skills in the workplace based on answers from the *Background questionnaire* in PIAAC (OECD, 2014b). Numeracy is in the framework of PIAAC described as:

> the ability to access, use, interpret, and communicate mathematical information and ideas, in order to engage in and manage the mathematical demands of a range of situations in adult life.
>
> (PIAAC numeracy expert group, 2009, p. 21)

The skills variable *numeracy* is derived from five background questions (see table 1). In the answers from the PIAAC background questionnaire, men report that they use numeracy activities in work (as they are conceptualized in PIAAC) more often than women (see table 4.5, OECD, 2013, p. 150). In most countries, workers with permanent contracts report that they apply numeracy skills more often than workers with temporary or short-term contracts (see table 4.14, OECD, 2013, p. 159). Clerical support workers, technicians, professionals, and managers report using numeracy more often than average. On the other hand, in occupations like machine operators and assemblers; craft and trades workers; service and sales, workers report use of numeracy skills less often.

From the background questionnaire (OECD, 2014b), PIAAC also reports results on use of numeracy at work from the individual countries participating in the survey. The following are a few pertinent results. One is that Sweden (together with Norway) reports the lowest average use of numeracy at work. Another result is that Sweden has a larger than average gender difference in the mean use of numeracy, although it is smaller than in the other Nordic countries. And, thirdly, Sweden reports an average level of use of problem solving at work on a par with the other Nordic countries.

Mathematics as part of workers' competence

Competence is in this paper understood as a wholeness which is seen as something other than *competency* which reflects a more fragmented perspective on knowing (Wedege, 2001). According to Wedege (2001), competence is always linked to a subject (person or institution) and it concerns a readiness for action and thought, based on knowledge, know-how, and attitudes/feelings. Furthermore, competence is a result of learning processes in both everyday practice and education and it is always linked to a specific situational context.

In analyses of workplace data, it is clear that one dimension of mathematics as part of workplace competence is the notion of *being critical* (see Askew, 2015, pp. 707–709). One example is the way that lorry loaders in a study by Björklund Boistrup & Gustafsson (2014) remained critical of the plans in a loading list set up by administrators, since the plans were not always possible to accomplish due to reasons of security, dimensions of goods, etc. In parts of the literature there is also an emphasis on respect for the complexity of mathematics in work (Askew, 2015; FitzSimons, 2002, 2014; Sträßer, 2015) and the role of mathematics in adults' life worlds (Henningsen, 2006; Wedege, 2001, 2013).

Data collection and analytical framework

The analyses for question 1 are based on data from PIAAC public use files (OECD, 2014d). The Swedish PIAAC dataset comprises 4469 persons. Tables are created with the SAS-system using the Swedish dataset prgswep1.sas7bdat. Variables are identified from the *International codebook PIAAC public use file methods and variables* (OECD, 2014a). Information on use of mathematics in work was contained in the six items GQ03_b–GQ03_h used by PIAAC to create the skills variable *numeracy*. We also included the item GQ01_h (reading of tables and graphics) from the literacy panel.

Table 1. *Items for the measuring of mathematics use at work*

G_Q03b:	How often – Calculate prices, costs or budgets
G_Q03c:	How often – Use or calculate fractions or percentages
G_Q03d:	How often – Use a calculator
G_Q03f:	How often – Prepare charts, graphs or tables
G_Q03g:	How often – Use simple algebra or formulas
G_Q03h:	How often – Use advanced math or statistics
G_Q01h:	How often – Read diagrams maps or schematics

In the PIAAC survey the two items ”simple algebra or formulas” and ”advanced math or statistics” were explained as follows:

> By simple algebra or formula, we mean a mathematical rule that enables us to find an unknown number or quantity, for example a rule for finding an area when knowing length and width, or for working out how much more time is needed to travel a certain distance if speed is reduced, and more advanced math or statistics such as calculus, complex algebra, trigonometry or use of regression techniques.
>
> (G_Q03g and G_Q03h, OECD, 2014c)

For question 2 the data consists of the same questions in the background questionnaire as were analysed for question 1, but here we focused on the *wordings* of the questions themselves. We discuss the findings in relation to the framework of PIAAC as described above. In the analysis for question 2 we have compared background questions of PIAAC drawing on the competence and performance models from Bernstein (2000; see also FitzSimons, 2002; Tsatsaroni & Evans, 2015). The two models are described here albeit, by necessity, briefly. Likewise, ours is not a complete sociological analysis, but can be seen as an initial analysis and discussion of what can be learned, or not, from results of background questions in a study like PIAAC. Bernstein (2000, p. 45) presents the discourse of *competence model* the following way:

> Pedagogic discourse issues in the form of projects, themes, ranges of experience, a group base, in which the acquirers apparently have a great

> measure of control over selection, sequence and pace. […] [The evaluation orientation has an emphasis on] the realisation of competences that acquirers already possess.

The competence model of Bernstein (2000) is coherent with how we described workplace competence in a previous section. Drawing on FitzSimons (2000), our assumption is that the competence model as a learning discourse is dominant within many workplaces. The discourse in this model is often in the form of *projects*, including a range of practical *experiences*, along with theoretical underpinning knowledge. The term project can here refer to everyday work-tasks where mathematical aspects are interwoven with others. The lorry loaders mentioned previously, drew on various knowing to accomplish the loading of a trailer (as an example of project at work), such as estimation of space, length, calculations, as well as knowledge about rules, logistics for the driver, and also about compression of the load, etc. In the competence model, learners (including adult workers) generally have *control* over the situation within the framing of the work-task. The *performance model* by Bernstein is quite different from the competence model:

> Pedagogic discourse here issues in the form of the specialisation of subjects, skills, procedures which are clearly marked with respect to form and function. […] [The evaluation orientation has an emphasis on] explicit texts. Acquirers have relatively less control over selection, sequence and pace. Acquirers texts (performances) are graded, and stratification displaces differences between acquirers. (Bernstein, 2000, p. 45)

The *performance model* of Bernstein is most often what characterises formal mathematics education (FitzSimons, 2002). In this model it is not about what a person can accomplish as part of a wholeness, but rather about *skills* that are *clearly marked with respect to form and function*. Learners do not have much *control* over the situation (e.g., in selection, sequence and pace) and, when it comes to evaluation, the focus is on what is missing rather than what is present.

Patterns concerning adults' mathematics in work

Table 2 shows the self-reported use of mathematics at work for the employed sector of the Swedish population 2012.

In table 2 *often* is defined as *at least once a week*. Use of a calculator is the most common activity at work reported, with more than half of the respondents answering that they use a calculator at least once a week and one third using it on a daily basis. One third of the respondents report that they use or calculate fractions or percentages, i.e. engage in some form of arithmetic, and one third report that they *often* read diagrams, maps or schematics at work. More than half of the respondents report that they *never* use simple algebra or formulas at

Table 2. *Self reported use of mathematics at work (%)*

How often	Calculate prices, costs or budgets	Use or calculate fractions or per-centages	Use a cal-culator	Read diagrams maps or schema-tics	Prepare charts graphs or tables	Use simple algebra or formu-las	Use advanced math or statistics
Never	45,7	34,8	21,8	30,3	58,0	52,9	81,9
Less than once a month	16,1	15,4	12,5	20,4	19,4	15,2	11,0
Less than once a week but at least once a month	12,1	12,1	13,1	15,8	12,5	10,3	3,9
At least once a week but not every day	12,8	18,4	23,8	18,8	7,6	11,5	2,3
Every day	13,4	19,4	28,8	14,6	2,4	10,2	0,9
Often*)	**26,2**	**37,8**	**52,6**	**33,4**	**10,0**	**21,7**	**3,2**
Often (men)	29,5	45,4	60,0	42,3	13,1	26,7	4,8
Often (women)	22,5	29,4	44,7	23,8	6,7	16,3	1,5

Note. * Often is defined as at least once a week. Source: PIAAC Public Use Files prgswep1.sas-7bdat, G_Q03b–h, G_Q01h

work (as explained previously). One respondent out of five uses simple algebra or formulas at least once a week; and four out of five of the respondents report that they never use more advanced mathematics or statistics, such as calculus, complex algebra, trigonometry or regression techniques. Three per cent use advanced mathematics at least once a week and less than one in ten prepare charts graphs or tables at least once a week.

To get a richer picture, we have examined how the answers on use of mathematics in work are related to *gender* (see table 2). For all mathematical activities men report a more extensive use of PIAAC-mathematics than women. One could surmise that the difference between men and women stemmed from the gender segregated labour market where men and women come from different areas of study and work in different sectors of the economy. A gender gap in favour of men using mathematics is found in almost all countries (OECD, 2013) but, as previously noted, Sweden has a larger than average gender difference in the mean use of PIAAC-mathematics, although it is smaller than in the other Nordic countries

We also analysed the reported use of mathematics in work in relation to answers on questions about *problem solving* at work. It is not possible to determine whether respondents use mathematics (simple or advanced) in problem solving from the PIAAC data, but it is possible to examine the extent to which respondents who state that they solve problems at work also indicate that they use mathematics (calculations not shown). Use of advanced mathematics and statistics does not seem to play a major role in either complex or simple problem solving. Among those who indicated that they often dealt with advanced problem solving, only 5.5 % in total used advanced mathematics at work and only one out of three uses simple mathematics. For simple problem solving the corresponding figures are 4.0 % who use advanced mathematics and only one in four use simple mathematics. Conversely, almost half of the respondents who reported that they often used advanced mathematics, rarely solved complicated problems at work. Self-reported use of advanced mathematics has thus, in the PIAAC study, only a weak link to problem solving.

PIAAC-mathematics: competence or performance

As noted above, the analysis for question 2 about the characteristics of the background questionnaire utilised a selected aspect of Bernstein's (2000) framework. Tsatsaroni and Evans (2015) conducted a related analysis on the items in the numeracy test, whereas we have not examined the items themselves, but only the background questions.

As we noted above, the competence model by Bernstein (2000) appears as the most relevant for many workplaces, where ”projects” in which different knowing is integrated into a wholeness are common and where experience and contextual knowledge are essential. Generally, adult workers have far more control of the situation and work task at hand than a student still in school. Following this, it could be expected that questions to adults about mathematics in work would share the characteristics of the very same competence model. However, this is not the case.

Looking at the PIAAC-definition again, we analysed this as vaguely reflecting a competence model (our insertions in italic): ”the ability to access, use, interpret, and communicate mathematical information and ideas, in order to engage in and manage the mathematical demands of *work-life as one of many contexts*” (OECD, 2013). This describes something very similar to Wedege's (2001) definition of competence (discussed above), and it is also close to the competence model of Bernstein (2000).

When analysing the background questionnaire we cannot find any questions that reflect mathematics (or numeracy) as part of complex workplace situations. More often the focus is on *separate, disconnected skills* which are vaguely described, such as calculating prices, using a calculator, preparing charts, or using simple algebra. There is no clear connection to *projects* such as those commonly found in working life; decision making is included but with only a focus on limited *procedures*. On the whole, our analysis finds that the background questions in PIAAC on the mathematics supposedly reflecting work actually fit more closely to Bernstein's (2000) (school mathematical) *performance model*.

Our findings from this limited analysis correspond to a great extent with those from the study by Tsatsaroni and Evans (2015), and hence we can conclude that both tasks and background questions fit the performance model of Bernstein (2000). Here we find a tension between the PIAAC performance model and the actual competence needed in work where mathematics knowing is just one aspect of any worker's workplace competence.

Finally, in this section we want to draw the attention to the name of the PIAAC study: Programme for the international assessment of adult competencies. The last word, *competencies*, may be misleading to, for example, policy makers since it has a resemblance to the word competence. On the contrary, competencies (*competency* in singular), as opposed to the term *competence*, are (drawing on Wedege, 2001) actually about the performance of skills and procedures, which align closely with the performance model of Bernstein (2000) rather than the competence model.

Concluding discussion

In this discussion we return to the aim of our study which was to investigate possibilities and constraints when using data from PIAAC 2012 with an interest in adults and mathematics in work. We emphasise that there are possibilities in such a reanalysis, especially for finding new areas for further research. In the study, patterns were revealed regarding adults' possible use of mathematics in work. One such pattern is the low number of respondents reporting use of PIAAC-mathematics in work, and where Sweden (together with Norway) reports the lowest average use. This can be compared to the findings of several qualitative studies. One example is the lorry loaders, described previously, who adopted various mathematics containing activities in their work (see e.g. Björklund Boistrup & Gustafsson, 2014).

In the background questionnaire, PIAAC includes the reading of diagrams, maps, or schematics in the literacy domain (OECD, 2014b). However, a number of the problems in the domain of mathematics in work are also concerned with the reading of graphs and tables. This might reflect an ambivalence in the PIAAC investigation, that the reading of graphs and tables is classified to be in both the literacy and the numeracy domains. Accordingly, an area of future research could be the reading of graphs and tables in school mathematics and/or language studies, and how this is connected to out-of-school contexts such as workplaces.

A noteworthy pattern arising from our analysis in question 1 concerns gender. Men generally reported a more extensive use of PIAAC-mathematics than women. We contend that this analysis of the responses to the PIAAC background questionnaire reveals a pattern, which really is important to investigate further.

The constraints in our reanalysis are clearly captured in our analysis for question two, on the characteristics of mathematics in work "according to PIAAC." In our findings we have described how limited the conceptualisation of "PIAAC-mathematics" is in comparison with the complexities that are part of actual work and of the competences of workers. We construed related constraints when we did the analysis of reported problem solving in the analysis for question one. As described in our findings, there was a weak link between reported problem solving at work and use of PIAAC-mathematics. We certainly do not claim that all problem solving is clearly based on mathematics. Drawing from research such as Björklund Boistrup and Gustafsson (2014), we do assume a stronger link than shown by PIAAC. However, since the questions posed in the background questionnaire – and in the items in the numeracy test (Tsatsaroni & Evans, 2015) – do not reflect the complexity of mathematics in work, and thereby not the actual realities of workplace contexts for the respondents, these results are what might be expected. More research is needed in order to problematize how investigations like PIAAC continue to maintain a limited view on adults and mathematics in contexts such as workplaces.

It would be a considerable challenge to construct a questionnaire which truly reflected a competence model of mathematics in work (which would actually mean higher validity (see Tsatsaroni & Evans, 2015)), rather than a performance model, while also meeting the demands of reliability. ICT creates the possibility to perform these kinds of international comparisons but the demands of measurement create a discourse where what is possible to measure becomes the most important consideration (and also act as a constraint), at least, in the case of this paper, for how mathematics in work is ”viewed.” A question worthy of further problematisation is whether the practices of large international studies like PIAAC are worth the costs in comparison to what might otherwise be gained.

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Orchestration of mathematical discussions drawing on students' computer-based work

Mats Brunström and Maria Fahlgren

Research points out the importance of following up students' work on computer-based tasks with whole-class discussions in which students play a central role. However, at the same time, research highlights the challenge for teachers in orchestrating such follow-up discussions. This paper examines whether an established model developed as guidance for teachers to orchestrate mathematical whole-class discussions (Stein, Engle, Smith & Hughes, 2008) could be useful in this educational setting. Students' written responses to two different tasks are the main data used to examine the model. The results indicate that the model has great potential to guide these follow-up discussions.

Despite the importance of following up students' mathematical work with computers in whole-class discussions (Bartolini Bussi & Mariotti, 2008; Kieran, Tanguay & Solares, 2012), few researchers have studied teachers' implementation of such follow-up discussions. Kieran, Tanguay, and Solares (2012) demonstrate the challenge for teachers to orchestrate follow-up discussions which take students' computer-based work into account. However, there is a large body of research focusing on teachers' orchestration of productive whole-class discussions within non-technology environments (Franke, Kazemi & Battey, 2007; Stein et al., 2008). For example, researchers have investigated ways of enhancing student engagement in whole-class discussions, i.e. make them more student-centred. Based on an extensive research review, Stein et al. (2008) propose a model consisting of five practices to support teachers in the orchestration of whole-class discussions using students' problem solving strategies as point of departure. The model aims to go beyond ”show and tell” by supporting teachers to align student strategies to key mathematical ideas. In Sweden, this model has been taken up in a significant way through the government CPD initiative for teachers of mathematics, and it has been used as a conceptual framework in studies focusing on teachers' orchestration of productive whole-class discussions (Larsson, 2015).

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This paper draws on a design research project, conducted by the authors of this paper in collaboration with four upper-secondary school teachers. The project aimed to investigate the implementation of short teaching units consisting of a researcher-designed computer-based lesson and a follow-up whole-class lesson devised by the class teacher. During the first lesson in each unit, students worked in pairs on task sequences designed for a piece of dynamic mathema-tics software, in this case *GeoGebra*. Besides developing students' proficiency with relevant GeoGebra tools, the tasks were designed to foster student reasoning concerning functions, particularly exponential functions (Brunström & Fahlgren, 2015). Findings concerning the teachers' orchestration of the follow-up lessons indicate that they typically were too teacher-centred to provide strong support for the development of student reasoning (Fahlgren, 2015). These findings encouraged us to consider how to support teachers in their planning and implementation of productive follow-up lessons.

Accordingly, the aim of this paper is to examine whether the model suggested by Stein et al. (2008) could be useful as guidance for teachers in orchestrating whole-class student-centered discussions based on students' previous computer-based work.

Theoretical background

There is a growing interest among researchers in studying ways of creating mathematical whole-class discussions in which students play a central role (Franke et al., 2007). Often, the purpose is to follow up students' previous work in pairs or in small groups and use it as a basis for a collective mathematical discussion in order to develop students' mathematical understanding (Stein et al., 2008). However, engaging students in mathematical discussions in which each student is given an opportunity to participate actively while simultaneously making certain that the intended mathematical direction of the lesson is followed is not an easy undertaking for teachers (Franke et al., 2007; Stein et al., 2008).

To support teachers in the orchestration of whole-class discussions while using students' work as a departure, Stein et al. (2008, p. 321) propose a model of five practices as a tool. These practices are:

(1) anticipating likely student responses to cognitively demanding mathematical tasks,

(2) monitoring students' responses to the tasks during the explore phase,

(3) selecting particular students to present their mathematical responses during the discuss-and-summarize phase,

(4) purposefully sequencing the student responses that will be displayed, and

(5) helping the class make mathematical connections between different students' responses and between students' responses and the key ideas.

According to Stein et al. (2008), the teacher has to be aware of as many different solution strategies as possible to be able to anticipate various student responses to a problem. Furthermore, they argue, it is important to predict incorrect solutions as well as correct. The first practice serves as a preparation for the second one, in which the teacher observes and (preferably) makes notes about students' work to get an overview of the various solution strategies in use (Stein et al., 2008). This overview, then, serves as guidance in the subsequent practice, where the teacher should select particular students or student groups to present their responses in the class. When selecting particular responses it is important to make sure that these responses are useful to illustrate and discuss relevant mathematical ideas (Stein et al., 2008). It is also important to bring common misconceptions to the fore as a basis for discussion. The fourth practice, then, concerns sequencing the student responses so that ”teachers can maximize the chances that their mathematical goals for the discussion will be achieved” (Stein et al., 2008, p. 329). In the last practice, the role of the teacher is to orchestrate a whole-class discussion where ”mathematical ideas are surfaced, contradictions exposed, and understandings developed or consolidated” (Stein et al., 2008, p. 333).

Method

As mentioned in the Introduction, this study is part of a broader research project investigating the implementation of short teaching units consisting of a computer-based lesson and a follow-up whole-class discussion. The project was conducted together with four upper-secondary school teachers, who were responsible for the classroom implementation of three teaching units in one class each. In total 85 first year students (age 15–17) participated in the project. Central to the main project was the design of task sequences, one for each computer-based lesson, in the format of worksheets intended for course *Mathematics 1b*.

The overall aim of the task sequences was to foster student reasoning concerning exponential functions. In the first task sequence, students are introduced to a real world context, a sunflower that grows 30 % each week. In the subsequent task sequences, students gradually meet more generic situations that require more abstract mathematical reasoning. The researchers were responsible for designing a draft of the worksheets before they were deliberately discussed and revised at joint meetings with the teachers before then implementing each teaching unit. This paper focuses on how to approach follow-up discussions in the last teaching unit.

As a basis for evaluating the suitability of the Stein et al. model for this type of follow-up lesson, empirical findings derived from students' written responses on the worksheets in the main project are used. These responses had been inductively analysed to identify different categories of student strategies. It is the results from this empirical analysis that provide the basis for the further pedagogical analysis constituting this study. Bearing in mind specific elements of the Stein et al. model, for it to be applicable to follow-up discussion of tasks, it is necessary for: (a) students' computer-based work to result in various student strategies appropriate to discuss and compare and possible to align to key mathematical ideas, (b) the sequencing of the strategies to be of importance, and (c) it to be possible to anticipate several strategies used by the students.

Student responses to two different types of task embedded in the last task sequence are used: description/explanation tasks and prediction tasks. In description/explanation tasks students are expected to describe/explain the outcome of a particular investigation. In prediction tasks students are supposed to first predict an outcome before testing it and then reflecting on the outcome in relation to the prediction. Responses from one class on one example of each type are used to examine the five practices in the model.

Furthermore, audio-recordings collected at the joint meetings with teachers provided some information regarding anticipations about possible student responses and behaviors. While examining the five practices, we will assume that teachers only can get information on student responses during the students' work on the tasks. Although the researchers observed all the teaching units, the focus during the computer-based lessons was not on how the teachers were monitoring students' responses to the tasks, i.e. the second practice in the Stein et al. model.

Results and discussion

For each example the categories of student responses identified in the main study are introduced followed by a discussion in the light of the five practices.

Example 1. Description/explanation tasks

Figure 1 introduces the first task selected for analysis. The analysis of student responses resulted in four categories. These categories are presented and illustrated by student quotations below. The ordering is not of importance at this stage.

(i) Responses focusing on the visual experience of movement in the x-direction.
”When you change the value of slider C the graph will move to the right or to the left”

> Now we will leave the example with the sunflowers and instead we will study the general exponential function, $f(x) = C \cdot a^x$, and how its graph is depending on the values of *C* and *a*.
>
> Set the slider *a* at 2.
>
> **3.** Drag the slider *C* so that the value of *C* varies. Describe in your own words how to see the value of *C* in the graph.

Figure 1. *Example of a description task*

(ii) Responses focusing on the visual experience of movement in the *y*-direction.
"You can see that the value varies because the point *A* [referring to a point on the graph] moves with the graph, *x* is the same but the value of *y* changes"

(iii) Responses focusing on the connection between the parameter *C* and the value on the *y*-axis (in the point of intersection).
"The value of *C* becomes the graph's value on the *y*-axis"

(iv) Responses focusing on the connection between the parameter *C* and the value on the *y*-axis (in the point of intersection), but at the same time claiming that the value of *C* does not affect the rate of change.
"*C* decides the starting value, that is, where the graph intersects the *y*-axis. No matter what the value of *C* is, the graph will increase at the same rate"

On reflection, there are three aspects that emerge from the categorization: local view (ii, iii, and iv) or global view (i and iv), framing in terms of vertical movement (ii) or horizontal movement (i), and attention to rate of change (iv).

Discussion of the first example in the light of the five practices

(1) Anticipating likely student responses to cognitively demanding mathematical tasks

While preparing this task at the planning meeting, responses belonging to category (ii) and (iii) were discussed. Concerning the responses belonging to category (i), we argue that it should not be too difficult to anticipate if the teachers "put themselves in the position of their students while doing the task" (Stein et al., 2008, p. 323). The only responses that we regard as surprising, and hence hard to predict, are the responses claiming that the value of *C* does not affect the rate of change. However, as Stein et al. (2008) point out, once the task has been used in class the teacher can add unexpected responses to their list of anticipations.

(2) Monitoring students' responses to the tasks during the explore phase

The students are supposed to work and discuss in pairs and to express their responses in writing. Hence, it might be possible for the teacher to identify ideas present in the lesson to set an agenda for later discussions both by listening to student discussions and by reading their written responses.

(3) Selecting particular students to present their mathematical responses during the discuss-and-summarize phase

We suggest selecting at least one representative of each of the four categories. Categories (i) and (ii) are important to discuss and compare because they provide examples showing how different visual focuses could result in different experiences, i.e. horizontal or vertical movements.

The responses belonging to category (iii) are important to emphasize and discuss since these responses pinpoint the main characteristic of the parameter C. Further, it might be worth displaying several student responses from category (iii) to discuss the choice of wording. Some students were expressing themselves in terms of "starting value", and some were expressing that the value of C can be seen on the y-axis. There were also two responses using the word "intersects", which could be used by the teacher to introduce the term *point of intersection*. The possibility to introduce important mathematical notations in follow-up lessons is emphasized in the literature (Bartolini Bussi & Mariotti, 2008; Kieran et al., 2012).

Moreover, it is important to display responses belonging to category (iv), making the misconception concerning the rate of change visible and discussed. This could initiate a discussion on how the value of C affects the way the value of the function changes. This could be an ideal opportunity to clarify the difference between rate of change and relative rate of change, i.e. that the relative rate of change remains the same while the rate of change is affected by the value of C.

When selecting particular students to call on as candidate to present each idea in the whole-class discussion, teachers can try to ensure that different students are given the opportunity to present their ideas from time to time (Stein et al., 2008).

(4) Purposefully sequencing the student responses that will be displayed

We claim that it is preferable to first display and compare responses belonging to categories (i) and (ii). This allows for discussions about how local and global views might affect the dynamical visual experiences. This discussion then provides a base for an elaboration on the more focused question "how to see the value of C in the graph". At this stage, it is appropriate to discuss responses from category (iii) since they are based on a local view focusing on the intersection with the y-axis. Also, we find it important to display category (iii) before category (iv) and in this way focus on and finish the ”intersection discussion” before entering the discussion on how the value of C affects the rate of change. Hence, we argue that the sequencing of student responses is important in this example.

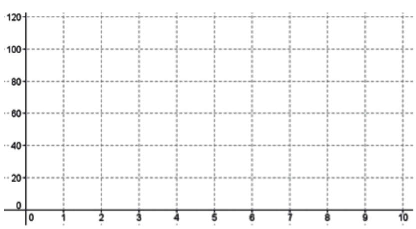
(5) Helping the class make mathematical connections between different students' responses and between students' responses and the key ideas

Although there was no request for an explanation in this task, the explanations concerning the key mathematical ideas, i.e. the intersection property and how the value of C affects the rate of change, are appropriate to discuss in the follow-up discussion. Preferably, the projected computer screen could serve as joint reference for students when presenting their various ideas.

Example 2. Prediction tasks

Figure 2 introduces the second task selected for analysis. Before this task, students had investigated how the parameter a affects the graph when $a > 1$ and $a = 1$ respectively. All students realized that the graph should include the point (0, 80) and that it should be decreasing, even if the shape of their decreasing graphs varied. The analysis of student responses resulted in four categories. These categories are presented and illustrated by student quotations below.

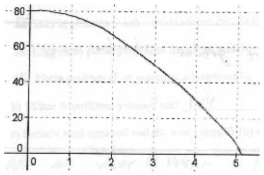
6 a) Guess (without using GeoGebra) what the graph of the function $f(x) = 80 \cdot 0{,}5^x$ will look like. Make a sketch in the coordinate system below.



b) Explain the thoughts behind your guess.
c) Compare your guess with the graph obtained in GeoGebra. Explain any differences that may occur!

Figure 2. *Example of a prediction task*

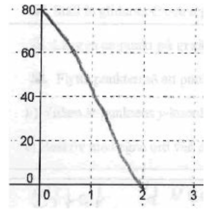
(i) Sketches showing a concave decreasing graph followed by an explanation comparing with the cases $a > 1$ and $a = 1$. Minor differences are expressed, when comparing with the graph obtained in GeoGebra.

a) 

b) ”Since 1 goes straight and 1.5 goes up, 0.5 must go down”

c) ”It lands at 8, but I thought that it should land at 5”

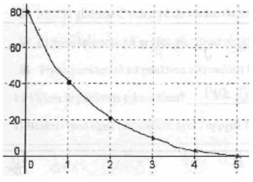
(ii) Sketches showing a decreasing straight line, followed by an explanation referring to the repeated halving or 50 % decrease of the value. Expressing that the guess is steeper, when comparing with the graph obtained in GeoGebra.

a) 

b) ”It will decrease by 50 % each week”

c) ”Ours is much steeper”

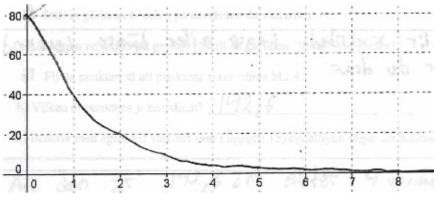
(iii) Sketches showing a convex decreasing graph crossing the x-axis, followed by an explanation referring to the repeated halving or 50 % decrease of the value. Expressing that their guess was consistent with the one obtained in GeoGebra.

a) 

b) ”The value is halved every time”

c) ”They look exactly the same”

(iv) Sketches showing a convex decreasing graph not crossing the x-axis, followed by an explanation referring to the repeated halving or 50 % decrease of the value. Expressing that their guess was consistent with the one obtained in GeoGebra.

a) 

b) ”Since it decreases by 0.5 the whole time from its new value”

c) ”Just right”

Besides the kind of explanations described above one student, in this class, claimed that "the higher the value of x is the less the decrease should be". To summarize, three of the four categories demonstrate student misconceptions (i, ii, and iii). These misconceptions concern the shape of the graph (i and ii) and/or intersection with the x-axis (i, ii, and iii). The explanations provided by students on task b) are of two types: comparing with the cases $a > 1$ and $a = 1$ (i) or referring to the "repeated halving or 50 % decrease" (ii, iii, and iv). Concerning task c), none of the participating students provided any explanation.

Discussion of the second example in the light of the five practices

(1) Anticipating likely student responses to cognitively demanding mathematical tasks

While discussing this task at the planning meeting, the teachers anticipated that most students would realize that the graph is decreasing, but that they would display sketches with different level of accuracy. The teachers anticipated that some students will make rough sketches based on a comparison with the cases $a > 1$ and $a = 1$, while others will insert points obtained by successively halving the starting value. When looking at the different kinds of graphs displayed by the students we claim that they all should be possible to anticipate, even if category (i) and category (ii) probably are the most difficult to predict. However, responses in category (i) are not that surprising, since these graphs could be obtained by reflecting a graph with $a > 1$ in the line obtained when $a = 1$. Concerning category (ii), research shows that students tend to revert to linear functions and that a common misconception is that if you have two points, a third point should lie on their linear extension (Kasmer & Kim, 2012). Category (iii) and category (iv) were the two dominant ones. However, it is not always easy to determine if a student response belongs to category (iii) or to category (iv). Therefore, it is extra important that these two categories are anticipated so that the teacher pays attention to the difference between them. The two types of explanation were anticipated at the planning meeting.

(2) Monitoring students' responses to the tasks during the explore phase

Concerning this practice, we refer to the discussion in relation to example 1.

(3) Selecting particular students to present their mathematical responses during the discuss-and-summarize phase

It became obvious from the student responses on this task that there is a need to clarify several things during the whole-class discussion. Responses belonging to category (i), (ii), and (iii) all contain serious misconceptions and it is clear from the responses on task c) that these misconceptions were not sorted out. Hence, we argue that it would be appropriate to select students so that category (i), (ii), and (iii) all are represented. Furthermore, (at least) one proper response belonging to category (iv) should be selected. Also, it could be instructive to select the explanation claiming that ”the higher the value of x, the lower the decrease should be”, since it pinpoints an important property of the graph.

(4) Purposefully sequencing the student responses that will be displayed

We suggest selecting a student response belonging to category (i) first. The reason for this is to first focus the discussion on the comparison between the various values of a, i.e. $a > 1$, $a = 1$ and $0 < a < 1$, and in this way focus on if the graph is increasing, constant or decreasing. In this way it will be possible to highlight the good thought behind this type of student response, before digging deeper into the shape of the graph. Next, a student response belonging to category (ii) might be a first step towards a more precise graph, since this will highlight the repeated halving or 50 % decrease of the value. The linear graph in category (ii) could be contrasted by the student statement ”the higher the value of x is the less should the decrease be”. Perhaps the student expressing this utterance also can explain why it is true. To make this clear, a student response using repeated halving or 50 % decrease to get values to be able to plot points in the coordinate system, like the one in category (iii), could be displayed. Finally, to focus the discussion on the question if the value becomes zero or not, a graph belonging to category (iii) could be contrasted by a graph belonging to category (iv). To summarise, we claim that it is important in which order the identified categories are displayed.

(5) Helping the class make mathematical connections between different students' responses and between students' responses and the key ideas

The variation in student responses makes it possible to pinpoint and discuss key ideas on the basis of some of these responses, not least responses revealing misconceptions. Important key ideas are: how the value of a determines if the graph is increasing, constant or decreasing, the difference between linearly decreasing and exponentially decreasing functions and that an exponentially decreasing function never intersects the x-axis.

Conclusion

Reflection on an observational study showing that whole-class discussions following up students' computer-based work tend to be teacher-centered (Fahlgren, 2015) raised the question of how to improve these discussions by making them more student-centered and tied to key mathematical ideas. The aim of this paper is to examine if the Stein et al. (2008) model for orchestrating productive mathematical whole-class discussions could be useful in following up students' computer-based work. Even if only two examples have been used to evaluate the model, these examples indicate that there is a great potential in the model also in this educational setting. In both examples we find all five practices in the model useful as guidance for teachers. However, more research is needed to examine the model further. Primarily the model has to be examined by observing classrooms where teachers are following a lesson plan based on the five practices.

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Student teachers' visions of good mathematics teaching and its (dis)connection to practice

Ola Helenius & Hanna Palmér

In this paper, three Swedish studies focusing on student teachers in transition from university to teacher practice are analyzed with respect to similarities and differences in how the teacher students describe the mathematics teaching they want to do as well as how they relate to teaching they already see carried out. Despite the different theoretical and methodological orientations in the examined studies, we find commonalities. One commonality is how the student teachers align with reform ideas when they talk about preferred mathematics teaching. Another commonality is how teaching observed in school based teacher education is typically described in negative terms since it does not conform to these reform ideas. We discuss this divide as a potentially negative effect of trying to use teacher education as a reform instrument.

Teacher education has a complicated relationship to school and to the teaching profession. On one hand, novice teachers or student teachers could naturally be seen as an apprentice in school practice who should "internalize and reproduce the norms that are characteristic for mathematics classes" (Jaworski & Gellert 2003, p. 847). On the other hand teacher education is often considered instrumental for school development and, at times, for school reform, which in Jaworski and Gellert's words is phrased as novice teacher's enactment of new or modified patterns of interaction.

A particular example of reform messages in mathematics education are the ideas popularized by the National Council of Teachers of Mathematics around 25 years ago (NCTM, 1989). This reform called for "a seismic shift away from a view of mathematics as the accumulation of rules and formulae which are drilled and practiced to one where mathematics is a sense-making activity and learners are actively engaged in their lessons" (Prescott & Cavanagh, 2008, p. 1). This reform message was influential in Sweden too where the national curriculum implemented 1994, LpO 94, was based on such ideas. In fact some of the influential scholars involved in NCTM's work, were also involved in the work leading up to the 1994 national curriculum in Sweden (Emanuelsson,

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Johansson & Lingefjärd 1992; Swedish National Agency for Education, 1994). However, as Kazemi and Loef-Franke (2004) notes, ”[A] large body of literature has demonstrated that supporting teachers to meet the ambitious and complex visions of mathematics reform is difficult” (p. 203). In line with these findings, in a relatively large scale qualitative study, it was shown that the 1994 national curriculum had not had much influence on the teaching, as far as some of the central tenets related to the reform movement go. Complementing the classroom research with teacher interviews and a psychological construct, the researchers' conclusion was that teachers had in general not understood the meaning of the reform message, probably due to a superficial interpretation of it (Boesen et al., 2014). Such superficial changes are characterized by a change in how one talks about the teaching and learning, but no corresponding change in ones beliefs or in the teaching practice (Ball, 1990; Charalambous & Philippou, 2010; Gregoire 2003).

One reason for the difficulty for teachers to meet the visions of the reform might be a mismatch between the visionary ideas about a new style of teaching and concrete descriptions of how to carry out such teaching (Kazemi & Loef-Franke, 2004). As noted by Skott (2004), recommendations within the reform sometimes degenerate into caricatures of what not to do (for example, not using whole class interaction or routine tasks) instead of focusing on what to do. This is in stark contrast to recommendations from research on teacher change, which for the last decade has emphasized the important connection between reflecting on new concepts or ideas and enacting those concepts with ones students (Clarke & Hollingsworth, 2002; Cordingly, Bell, Thomason & Firth, 2005; Grossman & McDonald, 2008).

In the light of this, it is interesting to ask in what sense the practice of teacher education affects students' views as far as reform ideas goes, and in what sense the teacher student are prepared to carry out these reform ideas in practical teaching. Adler writes

> Across the world, preservice and inservice mathematics teacher education programmes are preparing teachers to work with and promote reform in the practice of school mathematics. Although emphases will differ across the range of educational contexts, common threads are identifiable.
>
> (Adler 2000 p.205)

It is reasonable to assume that this occurs in Sweden too and that student teachers (at least superficially) adapt reform oriented ideas. But given the result shared above, (Boesen et al., 2014) it is perhaps also reasonable to assume that student teachers meet few examples of reform oriented teaching in their school based teacher education. Building on this, we will in this paper focus on the following two questions:

Q1. *Visions*: How do Swedish student teachers talk about teaching they would prefer to do? In particular, do they (at least superficially) align with the reform ideas?

Q2. *Practice*: How do Swedish student teachers talk about mathematics teaching they have seen or experienced, particularly in school based parts of teacher education?

The reason to investigate these questions is not primarily to understand in what sense student teachers become prepared to teach in line with reform ideas. Answering such a question would require different types of studies than this one. Rather, the combination of the two questions above puts a focus on the relation between teacher education and practice. Using novice teachers as implementers of new ideas is in line with considering teacher education as an instrument for school development. However, a strong divide between novice and experienced teachers would go against the idea of novice teachers and teacher students as apprentices.

No large-scale study has looked at these two questions in Sweden. However, there exist three different small-scale studies that, while working with different theories, methods and aims, all ask questions related to student teachers visions, goals and relation to school practice. In this paper, we make a secondary analysis of these studies to shed light on our two questions above.

Methodological concerns

The studies used in this paper are three Swedish theses of student teachers; Bjerneby Häll (2003), Persson (2009) and Palmér (2013).[1]

Bjerneby Häll studied ten respondents educated to become upper primary and lower secondary school mathematics teachers. The respondents were followed from the beginning of their teacher education and eight years forward with focus on how they formulated arguments for mathematics teaching in school. The empirical material was gathered through texts written by the respondents and through interviews.

Persson studied how 16 lower primary school teachers talked about mathematics teaching and how this talk changed throughout mathematics teacher education. The empirical material was gathered through interviews. Persson then continued to study how this talk changed after the respondents graduated and started to work as teachers.

Palmér studied the professional identity development of seven novice primary school mathematics teachers their first two years as novice teachers. The study began with interviews just before the students were to graduate from teacher education.

These three studies were selected since they are the published Swedish theses focused on the two questions raised in this paper. It is important to note that we are not trying to investigate knowledge, beliefs or change of teaching practice, only the results presented in the three studies. In the three theses there are several quotes from student teachers which have guided our analysis. However, based on space limitations only a few quotes will be re-produced in this paper. To answer the two questions we do not use any particular frameworks, constructs or theoretical perspectives. We claim this is in order, but below discuss some possible concerns with this method.

Interpretation of the reform

As explained above, LpO 94 consitutes a new national curriculum implemented in 1994, strongly connected to the type of reform mathematics teaching ideas that are commonly associated to the NTCM standards (Boesen at al., 2014). The three studies relate in different ways to the reform and the reform ideas. Palmér explicitly mentions much of the same literature that is referred to in the present paper. Bjerneby Häll, refers to principal ideas of Lpo94 (Swedish National Agency of Education, 1994) and especially focuses on using concrete materials (laborative mathematics) and variation in teaching, in most cases referring to not only using the text book. Persson does not write explicitly about reform in her thesis but about the "official style of thought" as expressed in steering documents in teacher education and primary school, which means that she implicitly refers to LpO 94. Based on these differences we cannot be particularly sure what aspects of the reform the student teachers relate to – if any. However, this is not a problem since our questions do not regard the process of teacher education and/or mathematics teaching but in what sense student teachers' talk about teaching (possibly superficially) align with the reform ideas.

Aggregation of results

The three studies have quite different theoretical perspectives. Bjerneby Häll relies on von Wright's practical reason and logic of events (1983) while Persson uses several perspectives, like for example Fleck's theory of thought styles (1935/1997). Palmér builds on Lave and Wenger's construct community of practice. In all three cases though, the theories are used to explain the participants reasoning or actions, not to structure the analysis of data in the first case. Since we pick up our data from the three studies on a more basic level, before the level of theoretically grounded explanations or argumentation, we draw the conclusion that it is possible to aggregate this data.

Results

It is eight years from the publication of the first to the last of these three studies, but it is 15 years between the first and last data collection. While the three studies focus on the transition from student teacher to teacher from different theoretical perspective, they all put significant emphasis on how student teachers throughout their university studies, change, adapt or construct their ideas about what constitutes good or favorable mathematics teaching. In all three studies it is also discussed how the student teachers related their personal ideas on mathematics teaching to teaching they have experienced before or are experiencing in the school based parts of their education (practice).

Q1. The student teachers' teaching visions

In the study by Bjerneby Häll, the teaching visions of five of the ten student teachers are exemplified by means of quotations. When the author summarizes, the views expressed by the student teachers become quite homogeneous. It is claimed that the teacher students have changed their views on mathematics teaching to a view in line with Lpo94. In summary the respondents express that they during teacher education have discovered that mathematics teaching can be laboratory, that the learning of mathematics will improve by communication and that mathematics problems can be solved in different ways. The respondents are critical of direct teaching; instead they stress a creative and exploratory approach in mathematics teaching. They say that they will use a text book when teaching mathematics but emphasize the importance of having a varied teaching approach not just teaching in line with the text book. Varied teaching is motivated as increasing the interest and motivation of the students to learn mathematics. The respondents also emphasize the importance of fun, understanding, self-esteem and laboratory elements in the lessons. Furthermore the mathematics teaching ought to be connected to everyday life. Only one exception from this is mentioned "[…] one informant (Ingrid) tells she does not remember anything from the courses in mathematics education" (p. 136).

Persson's study is divided in two parts where we here focus on how the student teachers in her study talk about mathematics and mathematics teaching before and after taking courses in mathematics education within teacher education. Before taking courses in mathematics education almost all of the sixteen respondents tell about memories of a mathematics teaching characterized by exercises in the text book. They have experienced this kind of teaching very differently where some liked it and others did not. After taking courses in mathematics education the respondents instead talk about their previously experienced mathematics teaching using words as *traditional* and *tradition* implying something negative. Before taking courses in mathematics education the student teachers talk more about the mathematical content than they do after taking courses. After the courses they talk about *how* the mathematics teaching is to be conducted and they say that they now understand that it is possible to teach mathematics differently than howthey were taught as students. Persson's results are univocal regardless of following the students as a collective or the individual change of each student. The student teachers express having received a changed understanding of the aim with mathematics teaching through teacher education. ”You have been talking about it a lot. We have discussed it and yes, you hear it in every lecture and you read it in every book. You almost become a little brainwashed I actually think” (p. 84). The preferred mathematics teaching is described as joyful, creative, and it should help the students to link the concrete to the abstract. It should involve practical problem solving and real-life situations.

Palmér's results are presented in both a chronological and a thematic way. In the chronological presentation four of the seven respondents are described individually while the thematic part is based on all seven respondents. Together these two parts present a joint picture of the seven respondents. Quotations from all respondents can be found in the thesis and it is clear, based on both quotations and author summaries that the student teachers have changed how they talk about mathematics teaching. The respondents use the words *traditional* and *old-fashioned* when they talk about less good mathematics teaching. Further, they consistently compare good and less good mathematics teaching with each other and often describe good mathematics teaching as ”teaching that doesn't …” followed by an example of less good mathematics teaching. The respondents express that they want to teach mathematics ”differently”. They say that they probably will use a textbook when teaching mathematics but emphasize the importance of having a varied teaching approach not just teaching in line with the textbook. Instead they talk about mathematics teaching that is reality based, creative, varied with for example laboratory elements and focused on processes. The students are to work a lot in groups, communicating, working with mathematics problems that can be solved in different ways. The respondents also emphasize the importance of students having fun and getting a good self-esteem. On one occasion, one of the respondents says that she does not agree with ”everything” but, apart from that, the respondents are very concurrent in their talk about the mathematics teaching they prefer.

In summary, aggregating over all three studies we draw the conclusion that all respondents, with the possible exception of just one, express a vision of mathematics teaching in line with the reform ideas.

Q2. Students thoughts on observed teaching

Turning to the question on how student teachers relate to mathematics teaching they have seen or experienced, particularly in school based parts of teacher education, the picture is again homogeneous. In Bjerneby Häll, nine out of ten student teachers say that the teaching they see in school based parts of teacher education differs little from the teaching they themselves experienced as students in school. While several have positive personal experience from their time as students, it is obvious that the changes in their ideas about mathematics teaching make them evaluate such teaching differently now. Bjerneby-Häll summarizes that the teacher students

> dissociate from the way of using the text book that they experienced as students or student teachers […] their mathematics lessons should not be as many of them experienced during school based teacher education, not only, ”take out the book and start to work”. (p. 155)

For five students there are explicit negative remarks about the school based parts of teacher education.

> They just followed the book straight down, chapter by chapter. My mentor wanted it like that. So I roughly did that in the groups I had. A few times I tried to use some more open tasks in grade 7. So that it wouldn't be so awfully boring. It didn't really turn out much like I wanted. You shouldn't care, really but you feel you should do like the mentor wants. (p. 143)

As mentioned above, the student teachers in Persson's study talk about their previously experienced mathematics teaching using words as *traditional* and *tradition* implying something negative after taking courses in mathematics. They express having met the same kind of teaching as they experienced as students during their school based parts of teacher education. They say that ”it [time] has stood still” (p. 87) and none of the respondents express the mathematics teaching they have met during practice as being in line with the mathematics teaching emphasized in teacher education. The student teachers express that there are no connections between teacher education and the mathematics teaching they have met in schools and that this make them uncertain as they think that it will take a lot more from them to teach in the new way. Some of the respondents also talk about the importance of them inspiring, and by that changing, the mathematics teaching of their future colleagues.

The student teachers in Palmér's study find it hard to give good examples from the teaching they have seen or experienced during their school based parts of teacher education. But, when they are asked to give examples of less good mathematics teaching ”[t]hen there is many” (p. 101). Often these examples are connected to the text book and teachers being ”very controlled by the text book” (p. 101) which, according to the student teachers makes the students ”finally think it is boring” (p. 101). ”[I have] been at two different schools quite a long time and it feels like many teachers are very controlled by the text book and that is what counts” (p. 99). The respondents' position away from their own experiences of mathematics teaching, even the ones who themselves experienced working in a text book as fun in school.

In summary, aggregating over all three studies we draw the conclusion that the student teachers in all three studies are critical of the teaching they have seen or experienced, both as students and in school based parts of teacher education.

Conclusion and discussion

The three studies together comprise over 30 student teachers, and the homogeneity in relation to our two questions is overwhelming. While we can still not reliably generalize to a larger population of student teachers, the different theoretical perspectives of the studies as well as the difference in time and place if anything strengthen the result.

A relevant observation is that it might not necessarily be because the student teachers know more about reform ideas that they grow skeptical towards the teaching they observe. As shown by Boesen et al. (2014), also the teacher practice tends to, in superficial words, be positive about reform ideas. What we have here, then, are two related practices that share a (possibly) superficial positive appraisal of reform ideas but do not share a practice where such ideas are actually enacted. The lack of practice is visible in the results presented above where the respondents can talk about good mathematics teaching but they have not experienced it, either as students themselves or during their practice periods. Quite the opposite, in the three studies the respondents talk about practice seems to be quite consistent with what Skott (2004) wrote about as caricatures of what not to do.

From a more general point of view, it must be considered problematic when student teachers hold such negative views of teaching in practice. Quantitative studies from the US based on a large scale experiments show experience lead to slightly better student performance (Nye, Konstantopoulos & Hedges, 2004). Other large scale studies in a German setting also indicate a positive effect on teacher knowledge from experience (Bauman et al., 2010). It is hence reasonable to assume that novice teacher have important things to learn from experienced teachers, as also indicated by the apprenticeship perspective in Jaworski and Gellert's model (2009).

Hemmi and Ryve (2015) have shown that the message regarding mathematics teaching in two Swedish teacher education programmes seemed to be quite homogenous. This message included several influences from reform movements. Based on their study they assume that student teachers' conceptions of good mathe-matics teaching are influenced by the homogenous message. The results presented in this paper indicate that Hemmi and Ryve's assumption is correct. While we do not know anything about the effectiveness of the student teachers in the three reviewed studies, what we can conclude is that the effect of teacher education has been a noticeable skepticism towards experienced teachers.

We end by noting that other studies too have found the relationship between mathematics teacher education in Sweden and teacher practice quite peculiar. Hegender (2009) observed that assessment of student teachers performance in the school based parts of education focused largely on relational, emotional and caring aspects of the profession, rather than mathematical knowledge for teaching. When comparing school based education from the point of view of university teachers, Ryve, Hemmi and Börjesson (2013) discovered that while Finnish educators saw it as a largely as a laboratory to carry out teaching, Swedish educators discussed it more as an organizational problem.

Together these studies raise some possible problems in the subject matter educational aspects of how teacher education in Sweden relate to experienced teachers and to practice in general. A systematic study of this relation could be a worthwhile future research effort.

Notes

1. Bjerneby Häll and Persson write in Swedish and all translations of citations are made by the authors of this paper.

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Learning mathematics: hope and despair

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To escape imposed deficit discourses around immigrant students when discussing their achievement in mathematics; this study examines immigrant students' own perspectives on their opportunities to learn mathematics. This was done by interviewing immigrant students from a multicultural and socially deprived area in Sweden, in two focus groups. In the interviews rowdy mathematics classrooms, the multicultural school and segregation emerged as hindrances that limit their opportunities to learn mathematics, creating a feeling of despair. However, the students demonstrated hope when talking about the future, which indicates a need for students to walk a balance between these two opposites when interpreting their opportunities to learn mathematics.

Until recently the Swedish school system has been characterized by the phrase, *a school for all*. However, during the last couple of decades differences in achievement between different groups of students within the Swedish school system have dramatically increased, indicating that ”a school for all” may not be the case (Tallberg Broman, 2014), particularly for immigrant students in socially deprived areas. According to Bunar (2009) multicultural schools in Sweden are often portrayed as bad schools with rowdy classrooms, a negative social climate and students with poor grades. This discourse is problematic since it leads young immigrant students' to believe not only that the opportunities to learn mathematics are limited, but also that attending a multicultural school is worse than going to a ”Swedish school” (Svensson, 2014). Deficit discourses of this nature are not uncommon in discussions of immigrant students' school achievement, not least because immigrant students are often construed, by those who are not immigrants, as ”problem students” who do poorly in school because they lack ”Swedishness” and have insufficient Swedish language skills (Dovemark, 2013; Norén, 2010; Runfors, 2003). Moreover, such deficit discourses are present in the Swedish media (Svensson et al., 2014) and, it is conjectured, likely to influence immigrant students negatively (Lange, 2008; Norén, 2010).

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To transfer from compulsory school to upper secondary school students must have a passing grade in Mathematics. However, despite a national picture showing the majority of students achieving this goal, there are schools in multicultural and socially deprived areas where only half of all the students do so (Skolverket, 2015). Consequently, there remain in Sweden many students who do not get access to the same social and academic opportunities in life as students from other areas in Sweden (Wigerfeldt, 2009).

Such matters of social justice and students' restricted life opportunities form one argument for this study. A second is to escape imposed deficit discourses by examining students' own perspectives on their situation. Therefore, the aim of this study is to discuss immigrant students' opportunities to learn mathematics by exploring how they describe their foreground, which here

> refers to a person's interpretation of his or her learning possibilities and "life" opportunities, in relation to what the socio-political context seems to make acceptable for and available to the person.
>
> (Alrø, Skovsmose & Valero, 2009, p. 7)

The guiding research question has been: How are the students' foregrounds expressed, and how do they contribute to the students' interpretations of their opportunities to learn mathematics? This article is derived from a licentiate study, investigating immigrant students' opportunities to learn mathematics (Svensson, 2014).

Theoretical background – foregrounds

One consequence of the deficit discourse is that individual students and their families are blamed for their failures in school and possibilities for change are limited since it is impossible to change a person's background. According to Skovsmose (1994) it is problematic to explain students' achievement with refe-rence to their backgrounds. He states that it is also important to consider the students' present situation and their foregrounds when explaining school achievement. Skovsmose (1994) states that intentions (here to learn mathematics) are grounded in a landscape of dispositions (pre-intentions) formed by, on the one hand, a person's background and, on the other, his or her foreground.

A person's background, drawing on the individual's history, comprises a socially constructed network of relationships and meanings. A person's foreground, on the other hand, reflects a "person's interpretation of his or her learning possibilities and 'life' opportunities, in relation to what the socio-political context seems to make acceptable for and available to the person" (Alrø, Skovsmose & Valero, 2009, p. 7) and can be expected to provide information about students' interpretations of their opportunities to learn mathematics (Alrø et al., 2009). In this way, foreground can be construed as a theoretical tool for analyzing, for example, student interviews. Both background and foreground are strongly connected to the individual, are interpreted by the person herself or himself, and cannot therefore be objectively viewed. A student cannot have one true foreground, since a student might have different foregrounds in different situations and foregrounds might also change over time (Alrø et al., 2009). According to Skovsmose (2014) a foreground includes both possibilities and obstructions and can be considered part of a student's life situation or, in Skovsmose's words, *a life-world*. A life-world is structured by economic, political, cultural and discursive factors wherein life-conditions or lived-through realities are a part (Skovsmose, 2014). He also points out that a foreground stretches beyond a life-world since it contains, for example, hopes, aspirations, wonderings, frustrations and despairs. It is also of importance to point out that through exploitation and stereotyping foregrounds can be imposed on a group of people (Skovsmose, 2012). In the particular context of this article, I have structured students' utterances into two foregrounded narratives, *hope* and *despair*. In so doing I highlight the relationship of two important foregrounds, as part of these students' life-worlds, to their future plans, the multicultural school and school mathematics and teaching.

Exploring foregrounds

To be able to learn more about how immigrant students, living in a multicultural and socially deprived area, perceive their opportunities to learn mathematics the study was undertaken at a compulsory school located in an area with a high immigrant population. The school had around 450 students from grade F to 9 and more or less all the students had foreign backgrounds.

In order to try to get a hold of the students' perspectives semi-structured life world interviews inspired by Kvale (1997) were conducted in two focus groups; two girls and one boy in the first and four boys in the second (see Svensson, 2014), shortly before the students were about to leave compulsory school, which means that they were either 15 or 16 years old. The interviews, approximately 55 minutes, were audio recorded and transcribed. The timing of the interviews meant that the students had already considered their future educational desires when choosing which National program to apply for in upper secondary school. The interviews were done in Swedish, but were translated and reported by the author.

One of the students, Khaled, was relatively newly arrived and had been in Sweden for 4 years. The remainder had been born in Sweden or immigrated to Sweden before starting compulsory school. These students' parents had immigrated from Afghanistan, China, Iran, Iraq, Kosovo, Kurdistan and Somalia.

When reading the transcripts concerning students' perspectives on their school and school mathematics, two narratives contrasting opportunities and obstructions could be discerned. Thus two result sections could be constructed by placing excerpts in the corresponding section, being the first step in the analysis process. The first section contains expressions around the students' futures, thus demonstrating *hope*. The second section contains expressions around school mathematics and the multicultural school, which contains possible obstructions for learning mathematics and demonstrates a feeling of *despair*. Hope and despair are part of the students' foregrounds (Skovsmose, 2012) and thus the second part of the analysis process was to explicitly explore what represents the students' foregrounds in the two narratives and how the foregrounds might contribute to their perceptions of their opportunities to learn mathematics. The data of analysis is the language in use.

Results

A narrative of hope – the future

The students, in talking about their plans for their futures, presented foregrounds that can be interpreted as positive and hopeful. One student, Chang, was certain he would achieve, without hindrance, his future plans of becoming a medical doctor. He had applied for the natural science program and said, ”ehh, the grades are good, I will get access, I got preliminary decision and I got in”. Chang was explicit about his future and expressed no worries or hindrances, demonstrating hope. This is slightly different from the other students who expressed plans for their futures at the same time as discussing possible hindrances, of which mathematics was one. For example, Ana, whose plan was to become a psychologist, said that she could have chosen the natural science program if it wasn't for mathematics – ”Look, I could have chosen natural science because of the chemistry, since the natural science subjects are quite simple” – but today she can't stand the mathematics:

Ana: I mean it is only maths that destroys everything. I don't want to have math G or math F [the lowest grade]

Jasmin: Become a kindergarten teacher.

Ana: Never in my life. I can't even stand the maths we have now.

Jasmin and Ana had applied for the social science program at a school in the city and Chang had applied to a natural science program at another school in the city. Chang and Jasmin had received preliminary decisions, but not Ana since she had not achieved a passing grade in mathematics when she applied and was therefore not qualified for upper secondary school.

Jasmin commented that she in ten years will, ”ehm, work and have one child”. Jasmin has no dreams or plans on what she would like to work with in the future, but says that she would like to work as an au pair. But she knows she will not be studying, ”No, then I won't study, I am sure”.

Khaled was in a similar situation to Ana and was hoping, at the end of his final semester in compulsory school, to receive a passing grade in mathematics and gain access to the applied program in upper secondary school. He had actually applied for the nursing care program since he wanted to become a policeman. When Khaled is asked if his dreams were realistic he answered, yes, but when this was followed up with a question about hindrances his response was short, ”maths”.

Hassan's desire was to attend the social science program but school mathematics seemed to cause him problems. His mathematics teacher had recommended him to apply for summer school in mathematics to get a chance to study more mathematics and in that way be able to get a passing grade. When it came to his dreams about the future Hassan said, ”finish studying, get a good job, living the life and then think about children” and when he was asked to explain what he meant by a good job he said:

Hassan: That is, I think that is, if I now get into that school it is the uniform professions and such, policeman or something.

Petra: Policeman is your dream?

Hassan: Yes.

When Hassan was asked about hindrances to his dream he said, ”maths”. Also Tarek had applied for the social science program but unlike Ana and Hassan Tarek seemed to be sure that he would get in. In ten years Tarek thought that he would be done with university and that he would work with computers and IT, maybe as a computer programmer. When asked if his future was realistic and if his goals achievable, he answered, yes of course, although possible hindrances for Tarek could be laziness. Just like Chang, Mohammed had applied for a natural science program at an upper secondary school:

Moham.: Natural science at P-school. I know I have difficulties in particular with math, but then I focus more, that is I will rather take the science-subjects, physics, chemistry and biology, but just in math my two brothers have studied it, or three, so they can help me.

Petra: Mm, will you get access with that?

Moham.: Yes, I believe so.

Petra: Yes.

Moham.: So maybe 95 percent I am in.

After finishing upper secondary school Mohammed dreamt about becoming a veterinarian. He also had a plan B for achieving his goals if he failed to get into Swedish veterinary education. He thought of his dreams as realistic and said, ”if you really want I think it is realistic”. When talking about possible hindrances for reaching his goals Mohammed says ”the largest hindrance then is maths”.

With the exception of Jasmin, attending upper secondary school and thereafter college or university was a part of all students' foregrounds. Becoming a medical doctor, psychologist, computer programmer, veterinarian or policeman were clearly expressed elements of their foregrounds. Through the discussion of their various ambitions, students clearly foregrounded hope. They also thought of their future plans as realistic and achievable, further foregrounding hope. However, when discussing possible hindrances to the achievement of those dreams, with the exception of Chang and Tarek, school mathematics seemed a barrier, a gatekeeper to further studies and future dreams (Skovsmose, 1994; Stinson 2004). Despite this, the students still thought of their future plans as realistic.

A narrative of despair

School mathematics

A part of the students' life-worlds was that they, to a varying extent, had to study mathematics in upper secondary school. This could be seen as positive or negative; positive for Chang, but less positive for the others. In Khaled's life-world mathematics was thought to have the capacity for destroying his future:

Khaled: But math destroys what it is called, a part of our future that is when you don't get a passing grade and can continue to upper secondary school you have to study in it and such, it is kind of hard.

He also commented that mathematics gave him a disgusting feeling, which he clarified by saying, ”that is, you know you are going to fail, that is a big chance that you won't make it, less of a chance that you make it”. Jasmin and Ana viewed mathematics as ”hard work”, with Ana adding that it was ”boring”. In their life-worlds mathematics gave them headaches:

Ana: So, I don't get anything.

Jasmin: Headache.

Ana: Yes, headache.

Petra: Headache, what does?

Ana: I think it is boring and you I sometimes have got really big headaches from math.

However, Jasmin said, ”but it is fun when you can do it. When you can't do it, you can go mad”, which Ana agreed with; ”I know, when you know something you go, oh, it is really fun, when you don't know, only math who wants to? Who even needs it?” For Mohammed mathematics was ”tough” and for Hassan it was ”hard work”; while Tarek said ”you have to struggle”. In contrast to the rest of the students Chang spoke positively about his mathematics experiences, as part of his life-world:

Chang: That is, I don't know. I get the feeling kind of so, yes math gets me forward, something like that or mine is to try to show that you are the best. I don't know, I get something like that when you say math. That is, I think it is like that, but anyhow it is something good for me.

In both interviews a wish for more mathematics teachers during mathematics class was expressed. According to Ana, Jasmin, Khaled and Tarek, they get insufficient help from their teacher and an extra teacher is needed. Khaled and Tarek commented that:

Khaled: So often we have one teacher so you don't receive a lot of help and I have only had one teacher during the whole semester.

Tarek: Yes, it is too few.

Khaled: I haven't received much help, just a little help so we need an extra teacher.

Tarek: One teacher is not enough, everyone don't get help.

In the other interview Jasmin said, "I think there should be, more math teachers, yes, like three". Ana agreed that this would give them help:

Ana: So that you really get the help you need so that you don't just sit there for the whole lesson and get stuck at such an assignment and you understand nothing, so you don't get help.

Petra: Has it been like that now then?

Jasmin: Yes.

The students' conditions in school and their relationship to school mathematics show somewhat shared life-worlds, Chang being an exception. For the rest of the students, school mathematics seemed to have a negative role in the creation of mostly negative feelings. The students believed that more teachers and more help were needed in the classroom, indicating a belief that they did not get enough help and leading to a discernible feeling of despair. These experiences, as part of the students' foregrounds, cannot be ignored and are likely to have a negative impact on their perceptions of their opportunities to learn mathematics.

The multicultural school

The students also talked about rowdy mathematics classrooms, which they believed affected their opportunities to learn mathematics, thus also forming part of their foregrounds:

Moham.: Several times when you are in the classroom, the teacher talks so it is like everyone talks, the teacher doesn't say anything, yourself you become so unconcentrated and distracted by other things.

Petra: You started to say something there about the environment? Tarek.

Tarek: It really affects a lot, kind of if everyone talk, yes.

According to the students in both interviews it was their classmates that caused the rowdiness. Mohammed expressed it as, ”yes, think so because many right now in our school they don't struggle, they just show up in class”. The reason for this, according to the students, is that they have given up on learning. In particular, Mohammed said, ”yes because, there are some that think I will still fail why should I try? There are some that are lazy and a lot more”. Ana believed that it was ”because maybe they think they can't do it and they think it is no point to try so they don't try and just mess around”.

At the end of interview when the students were asked if they had anything to add, before the recorder was switched off, Khaled commented:

Khaled: But also, they put immigrants in one school and most Swedes in another. It should be mixed, Swedes and immigrants; it shouldn't be as much chaos like it is in X-school, kind of in another school, a Swedish school not as much as it is here. It should be mixed, Swedes and immigrants.

Segregation was part of the students' foregrounds. Mohammed continued the conversation by saying ”yes, but it is just that. It feels more like all immigrants that have arrived, it feels more like they have isolated them here”. This is reinforced not only a sense of injustice but reinforced the belief that the multicultural school was a poor choice and a ”Swedish school” a better one. Mohammed and Khaled continued:

Moham.: Of course, the Swedish mathematics, everyone is not the same but if there were Swedes mixed then you should adjust to the Swedish language, it feels more like the municipality or state have done it on purpose. It is sometimes just that feeling you get.

Khaled: We are completely on one side and they are completely on another side, the immigrants on one side and then the Swedes completely on another side, we have less teachers in math class, they have more, two three of them in one class and get more help.

He continued.

Khaled: And they say money to the school, if we shall learn something so and education, then you have to have money, the municipality has to give money, kind of we don't have as much money as other schools have, we throw out teachers and such, so we get less education and everything.

The way the students described their school matches the public discourse around multicultural schools (see Bunar, 2009) and was a part of their life-worlds and thus their foregrounds. In their utterances the multicultural school emerged as a worse choice than a ”Swedish school”. That is, students thought that ”Swedish schools” provided better alternatives with respect to learning opportunities as found in an earlier study where students acknowledged a discourse about ”the need to be Swedish” and accepted, as desirable qualities, those of being Swedish (Svensson et al., 2014). This is similar to how the students in Dovemark's (2013) study adopted a discourse about ”weak immigrant groups” which made it possible for immigrant students to be aware of being perceived by others as negatively different. Also immigrant status, segregation and injustice emerged as hindrances for the students in their learning of mathematics, contributing to a feeling of despair among the students. These discourses seem to be a part of the students' foregrounds, which most likely influence their interpretations of their opportunities to learn mathematics and in this case most likely create obstacles to learning mathematics.

Concluding reflections

In this study two different narratives have been discerned, demonstrating students' foregrounds in relation to hope and despair, indicating a need for students to walk a balance between these two opposites when interpreting their foregrounds and thus their opportunities to learn mathematics. The latter, despair, contains foregrounds imposed (see Skovsmose, 2012) by others with the potential of undermining students' foregrounds since they are unable to influence those conditions (rowdy mathematics classrooms, immigrant status, segregation and injustice). A consequence of this might be serious learning obstacles that students themselves cannot easily overcome. However, the students' foregrounds in relation to their future plans contain hope and it is noteworthy that those students without a passing grade in mathematics still show confidence regarding their future. In contrast to the imposed foregrounds, the foregrounds related to the students' future are more easily redirected by the students themselves. This might give teachers and schools the opportunity to provide students with new opportunities by introducing other elements to their foregrounds, creating alternative foregrounds within which students might act. Therefore it is important to know more about students' foregrounds in order to redirect them in ways that would create more meaningful opportunities for learning mathematics.

In this paper I have tried to show how immigrant students are positioned in complex ways, with imposed foregrounds that need to be taken seriously by all those responsible for their education. It is unacceptable that immigrant students find themselves trapped in imposed foregrounds that influence their understanding of their opportunities to learn mathematics. I also argue that the use of foreground as a theoretical tool opens up for the examination of other factors impacting on immigrant students' opportunities to learn mathematics than those connected to the students themselves and their backgrounds. Such matters call for further research in order that we might not only gain a better understanding of immigrant students' foregrounds but also facilitate their seeing and acting upon alternative and more productive foregrounds with respect to their mathematical learning.

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Students' strategies to continue geometric number sequences

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Number sequences can be useful tools for teaching generalization, functions, or variables, for instance. Consequently, there are many studies that have studied students' perception of number sequences and the strategies used to continue those sequences. However, a large part of the studies have been using arithmetic or quadratic number sequences. In this paper we present a study of students' strategies to continue non-contextualized geometric number sequences. Interview data from 18 students in years 9 to 12 (age 15–19) (in Sweden) was analysed. Five qualitatively different strategies have been discerned in the data. These strategies are not completely overlapping the strategies previously described in literature.

Number sequences can be useful tools in mathematics education. Patterns and number sequences have been suggested to help students better understand the use of variables and to practice students' ability to generalize (Mason, 1996; Orton & Orton, 1999). Horton (2000) has suggested that students could benefit from number sequences when learning about linear and exponential models. To continue a given number sequences, i.e. to find the next number in a sequence of the type (2, 4, 6, 8, _) is a part of a generalization process. However, continuing a sequence and express generality verbally is easier than to describe the same thing using algebraic notation (Zazkis & Liljedahl, 2002). Frobisher and Threlfall (1999) claim the importance of students meeting sequences of different types in the mathematics classrooms. Number sequences could be of different types and are classified according to structure and regularity.

In a *repeating sequence* a particular unit is repeated as in, e.g. (1, 2, 3, 1, 2, 3, 1, 2, 3, …) where the unit (1, 2, 3) is reoccurring.

In an *arithmetic sequence* an element can be found by adding a constant term to the preceding element. The sequence (2, 4, 6, 8, 10, …) is arithmetical since each pair of consecutive numbers are separated by a constant term (in this case 2).

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A *quadratic sequence* is a type of sequence where the difference between elements is in itself an arithmetic number sequence. An example could be (2, 4, 8, 14, 22, …). This sequence is quadratic since the difference between two consecutive number elements is comprising an arithmetic sequence, in this case (2, 4, 6, 8, …).

In a *geometric sequence* each pair of consecutive numbers have the same ratio. Or, if we put it differently, an element in a geometric number sequence can be found by multiplying the previous element by a fixed number. An example of a geometric number sequence is (2, 4, 8, 16, 32, …). This is geometric since each pair of consecutive numbers are separated by a constant factor.

In addition to the different types, number sequences are being presented to students in different representations: Word problems, Visual (pictorial), Table, Geometric and Numeric (number sequence) (Ye, 2005).

The strategies students use to continue number sequences have been described in different studies. Stacey (1989) described four methods of using number patterns to solve problems: *counting method, difference method, whole-object method* and *the linear method*. She studied students aged 9–13. Hargreaves et al (1999) also included younger students. Their students were 7–11 years old, and perhaps consequently Hargreaves et al (1999) also found slightly different categories of strategies: *looking for differences between terms, looking for the difference between the differences, looking for multiplication tables, looking at the nature of the numbers, looking at the nature of differences and combining terms to make other terms*. Bishop (2000) described six distinct strategies students aged 12–15 used to continue number sequences: *modell, multiply, apply proportional reasoning, skip count/add, use an expression,* and *other*. Ekdahl's (2012) study focused on the different ways number patterns were discerned by students (9–11 years old). Six different categories were identified. In summary these categories were associated with the way a part (a number) or several parts (numbers) in the number sequence were related to each other or to the whole sequence, alternative to an extension of the given sequence. Classroom studies of repeating patterns have been discussed by e.g. Papic (2007) and Warren & Cooper (2006). Arithmetic number sequences are probably the most frequently used types in mathematics education research and have been used in a large number of studies of students' strategies to continue number sequences (Bishop, 2000; Ekdahl, 2012; Hargreaves, Threlfall, Frobisher & Shorrocks-Taylor, 1999; Lin & Yang, 2004; Stacey, 1989). Students' strategies when continuing quadratic sequences have been described in several previous studies (Ekdahl, 2012; Hargreaves et al, 1999; Lin & Yang, 2005).

Basically all of the studies mentioned above involve increasing arithmetic number sequences. Some authors separate between linear and non-linear patterns. They then compare arithmetic number sequences with quadratic number sequences. However, how do we know that this should be the distinction – that quadratic and geometric sequences (both non-linear) are solved with the same strategies? Are the strategies researchers have found students using for arithmetic and quadratic sequences qualitatively different from strategies used to continue geometric sequences or any other type of number sequence? This is what we would like to explore. Hence, the aim of this study is to describe the qualitatively different categories of strategies students use to continue geometric number sequences.

Method

In order to be able to find as many strategies as possible, in terms of strategies to continue number sequences, a screening test was designed. Seven groups (classes) of students were screened with the test in order to look for candidates to interview. The groups were chosen from different levels and different schools; two groups (about 50 students in total) in lower secondary school, year 9 (age 15–16), and five groups (78 students) in upper secondary school (Science programme), years 10 to 12 (age 16–19).

The screening test comprised three number sequences, one arithmetic (3, 5, 7, 9, _), one quadratic (1, 4, 9, 16, _) and one geometric (1, 3, 9, 27, _). The students were asked to explain (in writing) their strategies to continue the different sequences. Based on the written answers to the tests, 18 students were selected for interviews (8 students from the secondary school groups and 10 students from the high school groups). The selection (which student to interview) was made on basis of the written screening test in order to embrace as large a variation as possible in students' strategies. A large variation is crucial in order to describe the different strategies students use (Marton & Booth, 1997). We are aware that the interview situation in itself can influence the students' answers (Hunting, 1997). We also know that the way tasks are designed can affect students' strategies when they generalize number patterns (Chua & Hoyles, 2013; Samson & Schäfer, 2007). Therefore, particular care was taken to ensure that the students were given the opportunity to describe their strategies in any way they felt suitable (written or verbally), and the interviews were semi-structured due to this consideration.

The tasks given to the students during the interview were non-contextualized, meaning that they were given just as numbers on a paper. In Ye's (2005) vocabulary we have given the students the problems only in the format of numeric number sequences – not visual (pictorial). The number sequences given to the students during the interview were (2, 4, 8, _), (1, 4, 16, _), (_, 125, 625, 3125) and (2, 8, 26, _), respectively. Three of the sequences are increasing with a traditional blank in the end, and one, (_, 125, 625, 3125) is in practice a decreasing number sequence written in increasing form (increasing numbers to the right). The purpose of using different kinds of number sequences, not only geometric sequences, was to be able to include as many different strategies as possible. The particular sequences the students should evaluate were tested in pilot interviews in order to include as many and as divergent strategies as possible. The sequences were handed to the students one-by-one, each on a single paper. Each number sequence in the interview was presented with numbers separated by blank spaces and with a line at the missing number the students were expected to find. The students were asked to continue the sequences and to find a general expression to describe the sequences. The interviews were audio-taped and later verbatim transcribed. Each interview took about 20–45 minutes.

We were inspired by phenomenography (Marton, 1981) and the method used in Ekdahl (2012). She focused on the different ways in which students discerned different number sequences. However, in our analysis we focus on the strategies to solve number sequences and search for similarities and differences between the students' strategies and descriptions. The strategies found were categorized in accordance with Marton & Booth (1997) and emerged from the different strategies the students used. A more detailed description of the data collection method has previously been given in two project theses (Lindahl & Tegnefur, 2012; Lindahl & Tegnefur, 2013).

Results and discussion

We found five qualitatively different categories of strategies for students to continue the geometric number sequences. There is no particular hierarchic order between the different categories. In brief, the different categories we have found are: *Operating with each number separately, Looking for common factors, Looking at the nature of the number, Looking at the difference between numbers* and *Looking at the element and its place (index)*.

The categories comprise different strategies and can include a variation of different strategies, but with common features. The different strategies are described in detail below. Excerpts and examples from students are inserted in order to exemplify the strategies.

Operating with each number separately

What characterises strategies in this category is that it involves operations on a single particular number in order to find the next number in the sequence. These strategies appear to be focusing particularly on the numbers. Figure 1 shows an example where a specific operation (addition, multiplication) is applied for the entire sequence, but in order to find the next number the operation is applied only on the present number. In this example the student operates on the 2 to get a 4 and operates on a 4 to reach 8. Another example in the same category can be found in the immediate student response.

Interv.: Then we go for the next sequence [(1, 4, 16, _)].

Student: Then I would like to take 16 times 16 ... 32 [sic!], because ... Or, wait, it does not work. I thought 1 times 1, but it does not work because 4 times 4 is 16, but 1 times 1 is still not ... This was a bit more complicated ... 1 times ... Let's see if one can take 1 times 4. No, it does not work. Wait, yes, I would take 16 times 4, I guess ...

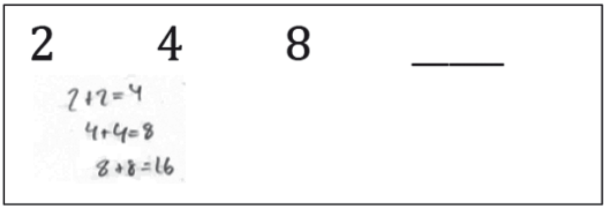

Figure 1. *In this case the student appears to operate with each number separately*

From the excerpt we see that the student's initial strategy is to operate with each number separately, in order to move from 16 to the next (unknown) number only 16 is operated on. One could argue that it is the square (4 x 4 = 16) that tricks the student to apply this strategy. However, as shown in figure 1 there are other situations too where strategies like this are tested.

These strategies are based on that the next number is sought by trying to apply a certain operation (to the sequence). But the operand(s) is limited to the present number. Hence, strategies in this category are based on operation with each number separately, not involving any other numbers or any difference between numbers.

Looking for common factors

Strategies in this category also focus on the numbers, but here a common factor is sought for in order to find the next number. The focus is not on each number separately, but on the collective property of the numbers in the sequence. In contrast to the previous category, here the multiplier (the factor) is the same for all numbers in the sequence. An example is shown in figure 2. Here the student shows the idea of looking for factors as the base in the powers.

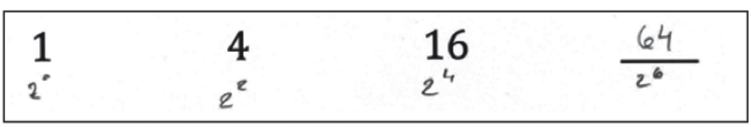

Figure 2. *The strategy seems in this case to be to look for a common base of the powers that constitute each number*

A related strategy is when the students for instance have shown that they look for factors to reach from one number to the next. As is shown in figure 3, the students could e.g. describe the operation on a number to find the next number. The operation, and particularly the operands, are more general, and not separate for each number. The strategies in this category do not include the multiplication

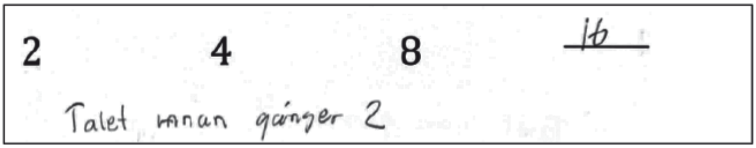

Figure 3. *This student strategy is described as "the number before times 2", which we interpret as if the student is looking at factors of the numbers in the sequence*

table. We consider looking for common factors to be qualitatively different from recognizing particular numbers from a multiplication table.

Both Hargreaves et al. (1999) and Stacey (1989) have described how students are looking for common features, like multiples. However, as they studied arithmetic and quadratic sequences and we study geometric sequences the common thing to look for must be different. In our case the students look for a common factor or a base in a power, whereas in their case the students can look for a common multiple. If we compare with the result by Ekdahl (2012) this category could be related to the perception of number sequences as related to equal motion between several parts of the sequence.

Looking at the nature of the number

Strategies in this category have in common that a particular nature of the numbers in the sequence is sought to find the next number. An example is the student that identifies a common property of the numbers in the sequence (_, 125, 625, 3125), as in this quote:

> It [the number] will probably end with 25, considering that it ends with 25 on all places, I would say.

In this category we can also find strategies based on trying to fit the numbers in the sequence into a particular multiplication table (although, not applicable to geometric sequences). A related example can be taken from one student trying to find a general formula for the sequence (2, 4, 8, _) expressed the following.

Student: There should be a 2 somewhere ...

Interv.: Why do you want a 2 here somewhere?

Student: Because 2 is twice as much ... Times 2 or power of 2 or something, maybe ...

A similar strategy can be seen in figure 4. The student has in this case sought for a formula including a number "2". In any case we consider the students efforts to be focusing on a common property or the collective nature of the numbers. We note that there is a wide span within this category. The properties of the number that is looked for can be of very different type. In the category *looking for the nature of the numbers* by Hargreaves et al (1999) they reported on properties such as odd and even numbers. In contrast, we observe other properties (here: that the numbers end with particular digits) being in focus when forming

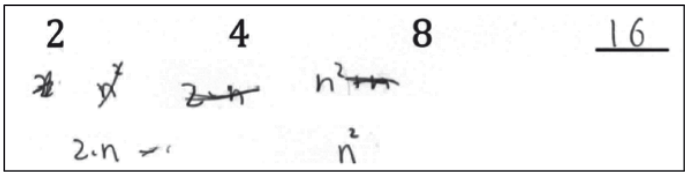

Figure 4. *The strategy seems in this case to look for known formulas all including a "2"; n^2 (written twice), $2+n$, n^2+n and $2 \times n$. The leftmost note could not be interpreted*

a strategy to find the next number. This could, however, be explained with that Hargreaves et al. (1999) studied younger students (age 7–11), whereas our study focuses on older students (age 15–19).

Looking at the difference between numbers

In this category the strategies are based on the fact that the difference between numbers is in focus. The difference is generalized in order to find the next number. An example is shown in figure 5. The strategy is explicitly stated as an operation with the differences between the numbers.

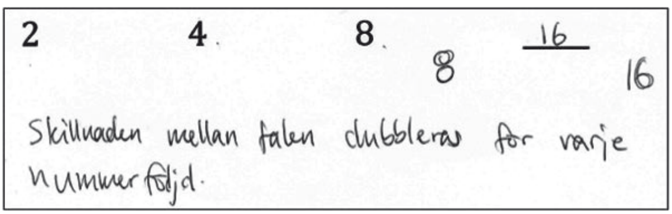

Figure 5. *This strategy is explicitly described as "the difference between the numbers is doubled for each number sequence"*

The strategy can also be found using powers, as in figure 6. In this example the student has shown the differences as repeated multiplication 2 x 1, 2 x 2, 2 x 2 x 2 and 2 x 2 x 2 x 2, respectively. We can compare this example with the strategy in figure 2. In that example the powers belong to the numbers, and in this example (figure 6) the powers (the repeated multiplication) belong solely to the difference between the numbers. As the given numbers are limited to three the sequences can be considered either as quadratic or as geometric. We observed that the sequence (2, 4, 8, _) gave particularly interesting data for the study. In this case there are two possible differences (4 – 2 = 2, and 8 – 4 = 4) and two possible ratios (4 / 2 = 2 and 8 / 4 = 2) to handle. Hence, if the strategy is to look at the difference we could anticipate either to look at the increasing difference (quadratic number sequence) or to look at the fixed ratio (geometric number sequence). This (quadratic/geometric duality) appears to be particularly evident in this number sequence. One student describes the duality as:

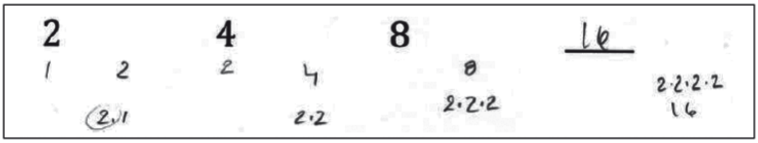

Figure 6. *In this example the student shows a strategy to find the generalization of the difference between the numbers*

Student: There it increases with 2 and then 4, then it should be 14

Interv.: Then it will be 14. Why should it be 14?

Student: Because here it increases with 2 and then ... or no, it can be both. It can be 16 too.

The operation on the difference can be of different types. Within this category we observe strategies where the differences are found through multiplication, addition or powers. However, we also observe a related strategy where the different differences are combined. In figure 7 is shown an example where the differences themselves are added.

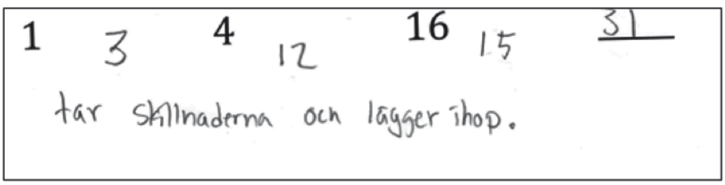

Figure 7. *The student has written "takes the differences and adds"*

Nevertheless, the common feature of the strategies in this category is that they are all based on an operation with what is not explicitly shown in the number sequences, but rather on what is in between what is shown – the differences between numbers. Both Ekdahl (2012) and Hargreaves et al (1999) describe the perception of what is between the numbers and points out the importance of the difference between numbers when generalizing number sequences.

Looking at the element and its place (index)

Strategies in this category look at the ordinal (index) of the element and its relation to the number at that place. A student shows an example of this strategy when trying to continue the sequence (2, 8, 26, _) by dividing 26 with its position or ordinal number (3). Similarly, another student tries to find a generalization by manipulating 16 (the third number) in the sequence (1, 4, 16, _) by operating on that number with 3 (number 16's ordinal number). Yet another student uses the strategy like this:

> Then I think one has to find something in common between these ... 2, 8, 26 ... and I am thinking that 8 is the same as 2 to the power of 3 and one

> should try to fit in some kind of order in the sequence too. Then it should be a one, a two, a three, a four, ... One cannot put *n* there either because if one changes 2, 8, 26 in these to a *n* it does not work either.

One could argue that the strategies based on looking at the element and its place is part of the category *Looking at the nature of the number*. However, in this case it is not really the inherent properties of the number that is in focus – it is the external property of its place in the number sequence that has a central role.

Discussion on the generality of strategies

Are the strategies students use different depending on whether we as researchers study geometric sequences or any other type of sequences? We note that the categories of strategies we have found have many similarities with, but do not completely map, the strategies by either Stacey (1989), Hargreaves et al (1999) or Bishop (2000). Particularly there are categories in Hargreaves et al's study on arithmetic and quadratic number sequences that are related to strategies we have identified. On the other hand, *Looking at the element and its place (index)* is not described in any of the studies on arithmetic sequences.

The representations we use for our number sequences are numerical which is different from the pictorial representation used by for instance Stacey (1989) and Bishop (2000). Actually, in many of the previous studies the patterns were given in visual (pictorial) representations. Possibly this could be a reason why we find slightly different strategies. Moreover, we do note that the students in our study is older than the students in previous studies. Mathematical experiences could play a role in the choice of strategy. However, we cannot completely rule out that the type of number sequence (arithmetic or geometric) can play a role in the difference we see in the strategies the students use compared to previous studies.

Conclusion

We have found five qualitatively different categories of strategies that students use to continue geometric number sequences. The categorized strategies have many similarities with strategies described in literature, but are not completely the same.

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Conceptualizing a local instruction theory in design research: report from a symposium

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This is a report on the discussions (and post-reflections) of the MADIF10 symposium ”Conceptualizing a local instructional theory in design research”. Linking the discussion to Koeno Gravemeijer’s keynote at MADIF9 and drawing on different ongoing research projects, the aim of the symposium was to discuss [examples of] the operationalization of design principles in order to deepen the understanding of some theoretical concepts in design research. The contribution of the symposium is the interpretation of how local instruction theory interrelates with other concepts in design research, for instance, the hypothetical learning trajectory. The role of the concepts as both design tools and as outcomes was presented and discussed.

Nowadays, *Educational design research* (i.e. ”design research”) is fairly often used as a methodological stance in doctoral and licentiate projects conducted in mathematics education research in Sweden. The reason for this may, on the one hand, depend on the typical professional background of the doctoral students, that is, in-service teaching. Hence, the research questions draw on a practice-driven realisation of the need to conduct studies aiming at improving teaching and learning in mathematics. On the other hand, research in mathematics education has also been criticized in the field for not producing useful instruction for teachers on how to design their teaching to improve learning (van den Akker, Keursten & Plomp, 1992; Reeves, 2006; Plomp, 2013), and educational design research is suggested as one way to develop such instruction. To this end, several research groups in Sweden apply design research methodology.

Reform mathematical pedagogy stresses inquiry and problematizing. This implies a change of perspective: from ready-made expert knowledge as a starting point for design to imagining students’ elaborating and refining their current way of knowing (e.g. describing the hypothetical learning trajectory?). What is needed is an instructional design that supports students in developing their reasoning towards more sophisticated mathematical reasoning (Gravemeijer,

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2004). Design research can be described as the development of a teaching intervention to solve a complex educational problem while generating advanced theoretical knowledge about these interventions and processes (Plomp, 2013). This is in contrast to comparing studies in certain contexts and coming to the conclusion that method A is better than method B (Reeves, 2006). Design research instead allows researchers to generate and test theories in classroom contexts.

There are two strands of studies in design research: validation studies and development studies (Plomp, 2013). Plomp describes development studies as studies with the purpose to develop research-based solutions to an educational problem, and, hence advancing the scholarly knowledge about the characteristics of the designed and evaluated interventions. Validation studies, on the other hand, aim to develop or validate theories on educational interventions. The studies presented at the symposium contain both these strands in their problem formulation. The features put forward in defining design research is the way it addresses practice-driven problems, and, moreover, the development of instructions based on local theories (Cobb, Confrey, Lehrer & Schauble, 2003).

The idea of the symposium developed through discussions among young researchers who plan to apply, or have applied, design research methodology in their studies. The discussion in the symposium drew on different projects in order to deepen the understanding of the concept local instruction theory (Gravemeijer & Cobb, 2013; Gravemeijer, 2015). We will therefore briefly describe the content of the presentations in the symposium in the next section.

The presentations

At the symposium, Mellroth presented her on-going doctoral project to develop and evaluate a professional development program for in-service teachers on how to teach and challenge mathematically highly able students in the ordinary classroom. Inclusive education and equity in student learning are important aspects to consider in the professional development program. To develop and evaluate the program design, research methodology is used and the expected results are two-folded, that is: the formulation of a research based local instruction theory for teacher learning in a professional development program, as well as a framework on which to base teachers' hypothetical learning trajectory when teaching mathematically highly able students in inclusive settings.

In planning a possible extension of a study within his doctoral project, Olsson has considered to use design research and he presented his plans in the symposium. The starting-point of his project is earlier studies (e.g. Granberg & Olsson, 2015) in which he has investigated task design aiming at engaging students in productive problem-solving and reasoning. The tasks were solved with the support of the dynamic software-program GeoGebra. Implementing the results in regular teaching would require further knowledge concerning, for instance, curricula, social norms, mathematical norms, and teaching preferences. At the symposium, Olsson explored if a possible way to implement the results in regular teaching could be through a design research project, that is, discussing possible endpoints, starting points, and an initial local instruction theory.

Liljekvist used design research methodology in her doctoral project where she and her colleagues developed and evaluated mathematical tasks (see Liljekvist, 2014). In the symposium, she problematized the process of transforming research results into classroom settings. The results of earlier studies on a micro level in a design cycle cannot be used without careful considerations. The need for an ecologically valid local instruction theory becomes evident, as we know, for instance, that teacher agency is an important aspect of successful interventions, and the importance to move beyond inefficient linear teaching instruction. She hence raised the issue of how to engage teachers in design research.

The report now continues with the theoretical concepts presented and discussed at the symposium. Then we describe how the developmental and validation aspects of design research is negotiated and established in the studies presented, that is, the different approaches to establishing a local instruction theory, including the theoretical grounds.

Developing a local instructional theory

Design research aims at understanding more of the interrelatedness between teaching and learning in order to improve teaching. Let us therefore link the symposium at MADIF10 to the constructivist stance explained by Koeno Gravemeijer in his keynote at MADIF9 in Umeå, 2014 (Gravemeijer, 2015). He raised the important questions (for researchers as well as for teachers) of what mathematics we want the students to construct, and, consequently, how we may design teaching that promotes students' construction of such mathematical knowledge, which is a main issue in both theory-building and practice development (see e.g. Cobb et al., 2003; Gravemeijer, 2004; Liljekvist, 2014; Ruthven & Goodchild, 2008).

> [...] if we want students to reinvent mathematics by doing mathematics, teachers have to adapt to how their students reason and help them build on their own thinking. To do so they need a framework of reference to base their HLTs on. We may offer them such frameworks in the form of ”local instruction theories” – and corresponding resources. A local instruction theory consists of theories about both the process of learning a specific topic and the means to support that learning. (Gravemeijer, 2015, p. 1)

In his keynote Gravemeijer pinpointed the specific kind of design research discussed here, that is, research for development of local instruction theories (Gravemeiljer, 2015). Such design research projects have three phases; 1) preparing for the experiment, 2) experimenting in the classroom, and 3) conducting retrospective analyses.

Preparing for the experiment means to formulate a local instruction theory that can be elaborated while conducting the design experiment (Gravemeijer & Cobb, 2014). The first step is to clarify endpoints formulated as mathematical learning goals, and thereafter consider a starting point. In this step, the researchers need to take into consideration the results of earlier instructions and problematize them beyond the purely mathematical goal. For instance, what is the need for changing classrooms norms and expectations of mathematical teaching? The local instruction theory hence includes conjectures of possible learning processes and possible means of supporting these learning processes.

In Olsson's study, this preparation phase is based on earlier research investigating mathematical reasoning and reasoning supported by dynamic software (see e.g. Granberg & Olsson, 2015, Lithner, 2008). For Mellroth, however, the preparation is slightly different as she needs to consider two-folded endpoints, that is, both on the teacher professional development level and the student level in classrooms. Gravemeijer & Cobb (2013) points out that theories emerging from design research are developed at various levels. At the level of the instructional activities *micro theories* are developed, *local instruction theories* are developed at the level of the teaching sequence (in the professional development course or in the classroom), and *domain specific theory* is developed as the umbrella of the two others. In Mellroth's study a micro theory can describe a specific activity, for example, teachers' role-play when solving a task, and in Olsson's study it may be a specific task aiming to be solved with the support of a dynamic software.

When endpoints, starting points, and the preliminary local instruction theory are formulated, the experimenting in the classroom or in the professional development course can start. One characteristic of a design experiment is the cyclic process of designing, testing and re-designing instructions. The researcher(s) conduct a thought experiment by envisioning how the proposed instructional activities might be realized, then analyse the actual process, and, finally consider refining specific aspects of the design before the next thought experiment, and so on. The retrospective analysis aims at contributing to the development of local instructions and more encompassing theories (Cobb, 2003).

Drawing on Simon (1995), Gravemeijer shows the need for researchers, as well as teachers, to have an idea of a possible path through instruction activities, that is, a *hypothetical learning trajectory* that supports the envisioning of students' thinking and learning (see e.g. Gravemeijer, 2015; Simon & Tzur, 2004). Based on to what extent the actual learning trajectory corresponds to the hypothetical one, new instructions and revised learning trajectories can be designed. Gravemeijer (2004) points out that neither teachers, nor researchers can rely on fixed teaching sequences, since a teacher continuously has to adapt to the actual thinking and learning of her students.

Instead, a teacher can be offered a framework with exemplary instructional activities as a source of inspiration, which was the aim of all the studies presented at the symposium. Creating hypothetical learning trajectories should be supported by local instruction theories as rationale for the description of envisioned learning. Local instruction theories will bridge between more general theories and the practice of how to create hypothetical learning trajectories. This is evident in Mellroth's study as she makes conjectures from hypothetical learning trajectories in the professional development course, as well as develop hypothetical learning trajectories *with* the teachers on their teaching in their classrooms.

Even if the design of local instructions is emphasized, there is a need to understand and conceptualize local preferences. Focus on local theories often means that design researchers develop frameworks that explain local circumstances where grand theories, such as Piaget´s development theories or Vygotskian theories of learning, are too general for clarifying local phenomena (Cobb et al., 2003; Liljekvist, 2014). The development of such intermediate frameworks may lead to proposals of alternative conceptions of the existing understanding of a domain, and this must be specified in terms of endpoints and possible trajectories for learning (Ruthven, Laborde, Leach & Tiberghien, 2009). Ruthven et al. suggest that focus on design tools provides an effective mechanism for developing teaching activities. In the next section we will hence give an overview of the operationalization of design principles presented at the symposium.

Operationalization of design principles

To develop instructions and design principles, there is a need of insights into how and what the students learn during a teaching sequence, that is, their learning trajectories. The local instruction theory constitutes a framework for developing hypothetical learning trajectories describing possible learning through activities.

Even if there is a lack of useful instructions on how to design teaching, there are at least some design principles that could be used as a basis for further research. Gravemeijer described in his Keynote at MADIF9 how he and his colleagues used Freudenthal's Theory on realistic mathematics education (RME) as a base for design (Gravemeijer, 2015). Ruthven et al. (2009) suggest that Brousseau´s Theory of didactical situations (TDS) will provide a tool for designing teaching sequences. TDS has been used by Liljekvist (2014) as a starting-point for task design principles, and Olsson is also planning his design research based on this theory. TDS refers to problem-solving tasks and teaching environment designed to put students in learning situations where they will construct new knowledge through adapting existing knowledge. This includes a process of devolution, that is, it is the student's responsibility to solve the problem although supported by relevant feedback on her actions (see e.g. Brousseau, 1997).

In Mellroth's study, the interventions are based on developed and researched practical examples from Australia (UNSW, 2014), for example, and Germany (Benölken, 2015). The design principles are hence based on knowledge from literature and practice where the components are linked to each other in a consistent way, which ensures validity regarding how to teach and challenges mathematically highly able students in the ordinary classroom. Inclusive education and equity in student learning are important principles for designing the teachers' hypothetical learning trajectories. In this study, the practicability was ensured because the participating teachers choose themselves to join the professional development program and were supported by their principals. Another way to ensure the feasibility connection is its connection to research-based practice used in other countries aiming to support mathematically highly able students (Fuchs & Käpnick, 2009; Nolte, 2012; UCONN, 2015).

> [C]oupling the creation of scholarly knowledge within the practice of researching with the creation of craft knowledge within the practice of teaching makes possible approaches to collaboration between researchers and teachers which can contribute to building a more systematic knowledge-base for teaching. (Ruthven & Goodchild, 2008, p. 584)

In design research, one can choose to involve teachers' craft knowledge more or less in a study. However, the design research methodology invites researchers to engage in how to link the practice of research with teachers' practice of teaching, thus contributing to a teaching knowledge-base with resulting local instruction theories. In the operationalization of the design principles in the studies discussed in the symposium, teachers' craft knowledge is considered in different ways. In Mellroth's study the teacher is both a "student" and a partner in constructing the local instruction theory. In Olsson's and Liljekvist's forthcoming studies, the teacher is the carrier of craft knowledge within the practice of teaching to calibrate the hypothetical learning trajectories.

Summary

The discussion in the symposium was drawn on different design research projects in order to deepen our understanding of the concept of *local instructional theory*, and, consequently, its interrelatedness to a *hypothetical learning trajectory*.

The idea of this symposium was developed through discussions among young researchers who plan to apply, or have applied, design research methodology in their studies. One argument for using design research as methodology in mathematics education research is to increase the relevance of research for practice and for educational policy (van den Akker, Gravemeijer, McKenney & Nieveen, 2006). In design research, there is scholarly knowledge in both theory and practice, because theoretical insights and practical solutions in real world context are developed simultaneously (McKenney & Reeves, 2013).

The symposium focused on one key concept connected to design research in mathematics education: the local instruction theory (Gravemeijer & Cobb, 2013, Gravemeijer, 2015). The aim of the symposium was to share our understanding of the concept, discuss our arguments for the conceptualization, and how to operationalize it in each study.

The research studies that formed the basis of this symposium are in different work-in-progress stages: from being in the planning stage to already completed design cycles. The symposium was hence an opportunity to look in-depth into parts of the design research methodology through the different studies.

The contribution of the symposium is the interpretation of how local instruction theory interrelates with other concepts in design research, for instance, the hypothetical learning trajectory. The role of the concepts as both design tools and as outcomes was presented and discussed.

In light of the discussion on the MADIF9 keynote by Koeno Gravemeier (2015), it was a suitable topic for a MADIF10 symposium. We hope that the fruitful and critical discussions in the symposium will continue in the MADIF community in order to deepen our understanding of design research methodology, and to further the discussion of design research studies to develop local instruction theories in the Swedish mathematics education context.

Acknowledgement

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Inferentialism – a social pragmatic perspective on conceptual teaching and understanding in mathematics

Per Nilsson, Maike Schindler and Abdel Seidou

In the symposium, we will discuss the perspective that *Inferentialism* (Brandom, 1994; 2000) offers on conceptual teaching and understanding in mathematics. Brandom criticizes the representationalist way to view knowledge as mental (object-like) representations, which are assumed to be more or less correct representations of objects in an extern reality (Bakhurst, 2011). A representationalist view implies a "topic-by-topic approach" for teaching in which concepts and calculation procedures are taught atomistically. It implies the idea of knowledge growth as a linear enterprise where teachers must initially define some basic concepts in order to be able to gradually introduce additional elements. It is also assumed that once students have learned the definitions and procedures, they will be able to solve mathematical tasks by applying what they have learned (Bakker & Derry, 2011). Our hypothesis is that teaching and understanding of mathematics benefit from reconceptualizing knowledge as inferentialist; instead of considering conceptual understanding to be fundamentally keyed on mental states of representations, it is proposed that knowledge is primitively an ability, the ability to navigate in the *web of reasons* (Brandom, 2000; Bakhurst, 2011).

Inferentialism demonstrates that grasping a concept is an activity that involves commitment to the inferences implicit in its use in a practice of giving and asking for reasons (Bakker & Derry, 2011).

> To grasp or understand [...] a concept is to have practical mastery over the inferences it is involved in – to know, in the practical sense of being able to distinguish, what follows from the applicability of a concept, and what it follows from. (Brandom, 2000, p. 48)

Brandom illustrates this inferentialist perspective on conceptual understanding by comparing the human responsiveness to the responsiveness of a thermostat. Brandom (2000, p. 162) asks:

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> What is the knower able to *do* that [...] the thermostat cannot? After all they may respond differentially to *just* the same range of stimuli [...]

Brandom's answer to this is that, in contrast to the thermostat, the knower is not only responsive to external stimuli. The knower, the concept user, is responsive to reasons, the knower is moved by its ability to search for reasons and follow reasons:

> The knower has the practical know-how to situate that response in a network of inferential relations – to tell what follows from something being [...] cold, what would be evidence for it, what would be incompatible with it, and so on. (Brandom, 2000, p. 162).

Humans understand why they turn on the heating, whereas the thermostat does not. Following this line of reason, Bakker and Derry (2011) point out the significance for education. In education, students are not supposed to only show right reactions on certain stimuli. Instead, they are supposed to know reasons, to understand what they are doing, and become intelligent concept users.

Privileging Inferentialism over Representationalism does not diminish the importance of representation, because evidently "there is an important representational dimension to concept use" (Brandom, 2000, p. 28). However, the meaning of representations is not pre-given. Representations gain their meaning in their role in reasoning. Representations, as is the case with concepts in general, should be distinguished and understood precisely by their inferential articulation, that is, in terms of the conditions under which one is justified in using the concepts and aware of the consequences of accepting them.

Conceptual holism becomes a direct consequence of conceptualizing the conceptual on behalf of reasoning:

> [...] grasping a concept involves mastering the properties of inferential moves that connect it to many other concepts: those whose applicability follows from the applicability of the concept in question, those from whose applicability the applicability of the target concept follows, those whose applicability precludes or is precluded by it. (Brandom, 1994, p. 89).

Brandom (1994) introduces the term "web of reasons" as a metaphor of this holistic view of understanding. Webs of reasons are social in nature. Responsiveness to reasons allows our actions and claims to be constrained by norms or rules rather than simply by nature. On this account, webs of reasons are cast in the social *game of giving and asking for reasons* (GoGAR) (Bakhurst, 2011) where a move in GoGAR "can justify other moves, be justified by still others, and that closes off or precludes still other moves" (Brandom, 2000, p. 162).

By accounting for how students contribute to the game of giving and asking for reasons, we are not only provided an instrument by which we can account for students' conceptual understanding in light of having practical mastery over the inferences constitutive for the web of reasons (Bransen, 2002). We are also provided an instrument by which we can discern individual differences in how students participate in and contribute to mathematical reasoning in the classroom.

Research projects

In the symposium, we will present and discuss three research projects that use Inferentialism as theoretical frame for conceptualizing learning and teaching mathematics. Per Nilsson presents a project, which aims at characterizing qualities in teaching mathematics for understanding by giving account of the inferences explicit or implicit in the social game of giving and asking for reasons. The project presented by Maike Schindler focuses on collaborative communication in students' inquiry-based group work. The analysis illustrates how meaning making occurs in collaborative communication, in which students show joint efforts in their meaning making. In the third presentation, Abdel Seidou connects GoGAR to experimentation-based teaching of the statistic concept *correlation*.

Mathematics, teaching and understanding – analysing the game of giving and asking for reasons in a classroom mathematical practice

(Per Nilsson)

Recent mathematics education reforms call for the instantiation of mathematics classroom environments where students have opportunities to develop their understandings in communicative and interactive classrooms (Staples, 2007 Hufferd-Ackles, Fuson & Sherin, 2004). This paper reports on how analytical constructs of Inferentialism (Brandom, 2000) can be used to account for teaching mathematics for understanding in whole-class discussion, which is based on students' group work.

In Inferentialism, conceptual meaning is tightly connected to patterns and norms of interaction. To give meaning to and to understand a concept is to account for how the concept is inferentially endorsed and used in the interactive game of giving and asking for reasons (Brandom, 2000).

Aim of study

The aim of the present paper is to characterize qualities in teaching mathematics for understanding by giving account of the inferences explicit or implicit in the social game of giving and asking for reasons.

Method

The data analysed in this study was gathered from a grade 6 (12–13 years old) classroom in Sweden. The episode that is analysed follows from group-work in which the students was discussing the task in figure 1.

How big part of the figure is dark? Write down two different fractions.

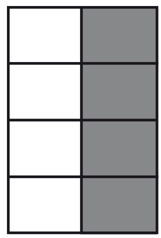

Figure 1. *The task*

The teacher of the class led the whole-class discussion. Moved by the game of giving and asking for reasons, the class develops inferential patterns related to the mathematical idea of *equivalent fractions*.

Collaborative meaning–making in inquiry–based learning: implications from Inferentialism

(Maike Schindler)

Inquiry-based learning (IBL) has generated much interest in mathematics education research (e.g. Pehkonen, 1997; Maaß & Artigue, 2013; Artigue & Blomhøj, 2013). With the project *Meaning–making in collaborative communication* (M2C2) I contribute to these efforts.

Based on Inferentialism (Brandom, 1994, 2000; Bakker & Derry, 2011) as background theory, I developed an analytical framework, which serves to evaluate group communication in mathematics according to the specific requirements in IBL practices: Collaborative communication, in which students make joint efforts in approaching problems and sharing ideas, is the focus of this study (Schindler, 2015).

In this paper, I present how the framework was used for analyzing collaborative meaning–making in an empirical study with students in upper secondary school; in a project, which addressed mathematically interested students from a Swedish gymnasium. In so-called *kreativa matteträffar*, which took place at the university, the students worked cooperatively on mathematical problems that were based on the ideas of Realistic mathematics education (RME, e.g. Gravemeijer & Bakker, 2006) and IBL.

The data analysis focuses on their collaborative meaning–making activities, in which they inquire and negotiate the mathematical content. I will present what factors contribute to students' collaborative meaning–making in IBL.

Aim of study and research questions

The present paper focuses on the questions: How does meaning–making occur in students' collaborative communication in IBL group work? What factors contribute to collaborative meaning–making in IBL?

Developing a local instruction theory for the learning of correlation in statistics – results and implications of a pilot study

(Abdel Seidou)

The present paper is a part of a larger project aiming at developing a Local instructional theory (LIT) (Gravemeijer, 2004) for the learning of statistical correlations. In this paper, we report on the results and implications of a pilot study.

The construction of the LIT builds on a *content dimension*, related to products and processes associated with statistical correlation; a *teaching dimension*, stressing collaborative teaching and students' experimentation with data and a *knowledge and learning perspective*, connected to the theory of Inferentialism and, particularly, to the *game of giving and asking for reason* (GoGAR) (Brandom, 1994; 2000).

Presentation and future research

During the symposium, we first present the results and implications of the pilot study indicating how discursive and organizational aspects of the task formulation and the activity setup affect how the concept of correlation come into

Figure 2. *Local instructional theory*

play in the GoGAR. For instance, the formulation of the task's question: ”*how* sure are you?” prompted students to a numerical answer without further elaboration, resulting in GoGAR poor of content related to correlation. Second, we set forth the set-up of the next step of the project. Based on the lessons learnt from the pilot study, the next step is to develop and conduct an empirical study based on the general principles of Design Experiment in educational research (Cobb, et al., 2003).

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A research program for studying the development and impact of formative assessment

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This paper outlines the research program for the formative assessment group at Umeå Mathematics Education Research Centre. The program was presented in a symposium at the conference, and focuses on the study of the development and impact of formative assessment. The main purpose of the research carried out by the research group is to provide research results that will be used outside the research community for educational decisions on systemic level, or as support for improved teaching and learning at classroom level. The paper outlines the fundamental ideas of the program, current studies, and examples of completed studies.

In 1998 Black and Wiliam published their influential review on the impact of formative assessment. They concluded that large-scale student achievement gains are possible when formative assessment is employed in classroom practice. This sparked a strong upsurge in the interest in formative assessment, and the number of published articles about formative assessment has grown dramatically during the 21st century (Hirsch & Lindberg, 2015). In addition, the significance of formative assessment for educational practice is emphasized by international organisations such as the Organisation for Economic Co-operation and Development (OECD, 2005). The number of studies about formative assessment in Sweden is also growing, but is still quite limited, both in general (Hirsh & Lindberg, 2015), and specifically in mathematics (Ryve et al., 2015).

Formative assessment can be defined in the following way:

> Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about next steps in instruction that are likely to be better, or be better founded, than the decisions they would have taken in the absence of evidence that was elicited.
>
> (Black & Wiliam, 2009, p. 9)

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This definition affords several foci for formative assessment, and Black and Wiliam's review, which was based on a shorter formulated definition with the same meaning, indeed included studies examining the impact of different strategies for formative assessment. However, the exact meaning of the concept differs between scholars. Some focus on the teacher using tests to gather evidence of student learning, with subsequent adjustment of instruction. Others focus on the feedback from the teachers, on the role students can play to support each other's learning, or on students' participation in the formative assessment process as self-regulated learners. Some scholars researching these strategies of formative assessment use the term formative assessment (or assessment for learning), while others use denotations specifying the specific focus, for example feedback. Research reviews focusing on each of these strategies have confirmed their potential for enhancing student achievement. The reviews include feedback (Hattie & Timperley, 2007), self-regulated learning (Dignath & Büttner, 2008), self-assessment using rubrics (Panadero & Jönsson, 2013), and peer-assisted learning (Rohrbeck et al., 2003). Research reviews focusing mathematics have shown strong relationships between student achievement, and teachers' adjustment of teaching based on collected evidence of student learning (National Mathematics Advisory Panel, 2008; Yeh, 2009) and self-regulated learning (Dignath & Büttner, 2008).

However, a strong research base supporting how to effectively help teachers to implement a high quality formative assessment practice is lacking (Schneider & Randel, 2010; Wiliam, 2010), and many professional development initiatives have been unsuccessful in accomplishing a substantially developed formative assessment practice to the extent that increased student achievement was obtained (Randel et al., 2011; Schneider & Randel, 2010).

The different strategies of formative assessment above share a core of modifying teaching and learning based on identified student learning needs, but focus on different aspects of formative assessment. Thus, a classroom practice that integrates these strategies into a unity could open up extended learning opportunities. However, such a practice would be more complex and provide further difficulties in its implementation. Suggestions for such conceptualisations exist (e.g. Wiliam & Thompson, 2008). However, even though some successful attempts have been made with a random selection of teachers (Wiliam, Lee, Harrison & Black, 2004; Andersson, 2015), studies provide evidence on the difficulty of supporting teachers to developing such a formative assessment practice to the extent that it significantly affects student achievement (Bell et al., 2008; Randel et al., 2011).

The research program

The research group in formative assessment at Umeå University currently includes 10 researchers from Umeå Mathematics Education Research Centre

(UMERC). The main purpose of the research carried out is to provide research results that will be used outside the research community for educational decisions on systemic level, or as support for improved teaching and learning at classroom level. The choices and design of research projects reflects the desire to achieve the goal of being in direct service to the education community.

Therefore, the group has a strategy to engage in combined research and school development projects, in which we collaborate with schools and municipalities for mutual benefit. A main type of research carried out by the group includes designing professional development programs, and studying the significance of characteristics of such programs for outcomes such as teachers' development of a formative classroom practice and student learning. Research also focuses on the relation between characteristics of formative assessment and student outcomes. Current studies conducted by the research group include (1) a three-year combined research and school development project in an upper-secondary school based on professional development in formative assessment conceptualised as a unity of integrated strategies, and (2) a study of the impact of improved teacher support for students' self-regulated learning on students' mathematics learning activities in the early school years.

As a complement to such developmental research, we are also engaged in another type of cooperation with a municipality. In this project we study the impact of a professional development program (PDP) in formative assessment on mathematics teachers' practice. This PDP is organised by the municipality itself and is carried out in all their schools at compulsory level. Such research is conducted to gain research insights about implementations made with the intent to improve teaching and learning, and the results are intended to be used in subsequent professional development initiatives to improve the support to teachers. Another kind of research we carry out, as a complement to developmental research in collaboration with schools and municipalities, is laboratory studies about the impact of different types of reasoning on student achievement and how these types can be supported by formative feedback. These studies are made in collaboration with the UMERC research group on mathematical reasoning. The results of these studies are intended to be used in upcoming school developmental projects. For the same reason, the writing of research reviews is another complement to the developmental research that is the main focus of the group's research activities. A review of the impact of different approaches for formative assessment on student mathematics achievement is completed and currently under review, and a review on the impact of formative feedback on different types of mathematical reasoning is in progress.

The focus of our previous studies has been on mathematics, and this subject will continue to be of special importance in future studies. In addition, current studies also include other subjects, as we now conduct research projects involving all teachers and subjects in whole schools. Some studies focus on feedback or self-regulated learning, which aim at a specific aspect of formative assessment, but in most of our studies the content includes strategies for several aspects of formative assessment.

Examples of completed studies in the research program

In the following we describe some recently completed studies in a research project about the effects of a teacher professional development program in formative assessment we developed. In the first of these studies the teaching practice of a random selection of mathematics teachers was analysed before they entered the program. The specific aim was to investigate how the teachers used formative assessment. This is of importance since little is known about Swedish mathematics teachers' current use of formative assessment (Ryve et al., 2015), and thus about the possible value of, and specific content to include in, professional development programs in formative assessment.

The same teachers were then freed from teaching for 20% during one term for participating in the professional development program (PDP). The following school year they went back to normal teaching loads again. We examined the impact of the PDP on the teachers' practice and their students' achievement in mathematics, as well as the reasons for the type of changes the teachers made in their classroom practice due to the PDP.

In a follow-up study an in-depth analysis is provided of the knowledge and skills used by one of the year 4-teachers when applying formative assessment principles. At the heart of definitions of formative assessment lies the idea of collecting evidence of students' learning, and using this information to modify teaching and learning to better meet students' learning needs. Such regulation of learning processes would require skills to elicit the thinking underlying students' oral and written responses, and the capacity to make suitable instructional decisions based on this thinking. Sufficient knowledge about the character and use of mathematics teachers' knowledge and skills when practicing formative assessment is lacking (Heritage, Kim, Vendlinski & Herman, 2009). The aim of this study is to characterize the knowledge and skills that the teacher uses in her formative assessment practice during whole-class sessions.

Methods

A framework for operationalization of formative assessment conceptualised as a unity of integrated strategies by Wiliam and Thompson (2008) was used both for the development of the PDP, and for the analysis of teachers' practice. The framework comprises a big idea of using assessment to identify student learning needs and modifying teaching to meet these needs. As a complement it includes five key strategies involving the teacher and students in the processes of identifying the learning goals, the students' learning, and how to take the next step in the learning towards the goals. The key strategies are (1) clarifying learning intentions and criteria for success, (2) engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding, (3) providing feedback that moves learners forward, (4) activating students as instructional resources for one another, and (5) activating students as the owners of their own learning.

To examine teachers' formative assessment practice before the PDP two random samples of mathematics teachers from a mid-sized Swedish municipality were analysed (21 teachers teaching school year 4, and 17 teachers teaching year 7). The teachers were interviewed and their classroom practices were observed twice. The interviews and observations were semi-structured and interview guides and observation schemes directed the data collection. The analysis was guided by the Wiliam and Thompson framework. The purpose with the analysis was to identify the activities of the teachers' classroom practice that were used regularly and could be regarded as formative assessment.

To investigate the impact of the program on the teachers' practice interviews with the teachers about their changes in practice were made in the end of the school year that followed the PDP. In addition, complementary data were collected through unannounced classroom observations during this school year. Data about the teachers' practice before the PDP was already available from the first study. Using the framework by Wiliam and Thompson the analysis of the data was carried out with the purpose of identifying the characteristics of the teachers' changes in their formative assessment practice. To collect additional data for the study about the reasons for the type of changes the teachers made in their practice, teacher questionnaires were administered immediately after the PDP and in the end of the school year following the PDP.

To study the impact of the professional development on student achievement, control groups were used. For both school year 4 and 7, all teachers not randomly selected to participate in the PDP constituted the control groups. To compare the increase in achievement both the students to the teachers in the intervention groups and the students to the teachers in the control groups took a mathematics pretest in the beginning of the school year after the PDP, and a posttest in the end of the same school year.

The teacher chosen for in-depth analysis was one of the year 4-teachers who had made significant changes in her teaching towards a more formative assessment practice. Mathematics lessons were observed and audio-recorded for 2 months. A number of episodes involving formative assessment were analysed. The analysis was carried out in three steps. First, the interactions between teacher and students were assessed as formative if existence of the three phases; eliciting, interpreting, and use of information were identified. Second, the teacher's actions during the phases were described. Finally, the knowledge and skills the teacher used were characterized. The definition of formative assessment above by Black and Wiliam (2009) was used as an analytic tool to identify the formative practice. A framework based on Shulman (1986), and Ball, Thames and Phelps (2008) was used to characterize knowledge and skills used by the teacher.

Results

The results of the study about the characteristics of the teachers' formative assessment practice before the PDP show that all teachers used formative assessment to some extent in their classrooms. Together they performed activities within all five key strategies and had different ways of adjusting instruction based on the information they collected about student learning. The study also identifies the characteristics of this practice, and the results indicate that there were relatively small differences in the classroom practice of year 4 and year 7 concerning formative assessment. However, it is clear from the study that there is much room for improvement in both quality and quantity of the formative assessment practice, and the study points to potential areas of development.

The results of the studies of the impact of the professional development program on the year 4 teachers and their students show that the PDP motivated the teachers to make large changes in their teaching. They added new formative assessment activities into their classroom practice to a level that had significant impact on student achievement in mathematics. The classes taught by the teachers who had participated in the PDP improved their achievement more than the classes in the control group, and this difference was statistically significant.

All teachers had implemented some of the formative assessment activities presented in the PDP, modifications of these or modifications of previously used activities. The teachers' changes span from complementing previous teaching with new activities that enhance the big idea in formative assessment to a classroom practice that is radically developed in its very foundation. None of the teachers seem to have only implemented an instrumental use of new formative assessment activities, which have been reported in several other studies (e.g. James & McCormick, 2009). Based on Wiliam and Thompson's framework (2008) further analysis shows that the teachers had developed their formative assessment practice in three dimensions: (1) the processes in teaching and learning of identifying the learning goals, the students' learning, and how to take the next step towards the goals, (2) agents in the classroom, and (3) the time from assessment to modification of teaching and learning. This three-dimensional development may have afforded new opportunities for student learning. First, the integration of the three key processes of teaching and learning may enhance student learning. Strengthening one of the processes improve the combined value of using them together. The second dimension indicates that further learning opportunities may occur by involving all agents (teacher, student, and peers) in the assessment process. The teacher and students work together to support learning through interaction during all three learning processes and the quality of students' support to each other and students' self-regulated learning can be improved. Lastly, shortened time between assessment and modification makes formative assessment more time efficient. Less time is spent on activities less optimal for the learning and less time is spent on waiting for help from the teacher, since the students are less dependent on the teacher.

Results also show that the reasons for the teachers' implementation of formative assessment activities were well explained by the expectancy-value theory of achievement motivation (Wigfield & Eccles, 2000). The teachers developed high value beliefs for the outcome of formative classroom practice as well as high expectancies of success to be able to carry out this kind of teaching. The value beliefs included for example high experienced utility value for both themselves and for the students, and only moderate costs in terms of time and effort. According to expectancy-value theory these variables are decisive for the motivation of action. Identified important aspects of the professional development program that motivated the teachers were: (1) A formative and process-oriented character, (2) activities directly useable in classrooms, (3) positive experience of using formative assessment activities, (4) connection between theory and practice, (5) time, and (6) knowledgeable support.

Similar to the studies about the year 4-teachers, preliminary results show that after the PDP all year 7-teachers were also highly motivated to develop their practice. They also did do so, but in different ways and to different degrees. The most common and frequent change was that the teachers more often, and in a structured way, elicited evidence of all students' learning with the purpose of adjusting their instruction (Key strategy 2), which led to more well-founded and more frequent adjustments of their teaching. Another common change was that they used more effective activities to engage and create thinking among all students during whole-class sessions. Only small or moderate changes were related to Key strategy 4 (peer-assisted learning) and Key strategy 5 (students as self-regulated learners). Thus, much of the responsibility for the formative classroom practice was still on the teachers. The analysis of the impact of the changes in teaching on student achievement has not yet been completed.

A main conclusion from the in-depth analysis of one of the year 4-teachers is that the formative assessment practice is a very complex, demanding and difficult task for the teacher in several ways. The analysis identified 13 activities the teacher used in the formative practice. Six of those formed the base of the teacher's formative assessment. These activities included the use of all-response systems, random selection of students to answer questions and the use of extended time to think. The teacher also engaged the students in taking an active part in the formative assessment practice. For instance, the students were asked to give examples of how to write fractions equal to 3/2. They gave their answers on their miniwhiteboards (an all-response system) so the teacher could receive information about all students' understanding. The teacher noted that not all of the suggestions were correct and decided to write the students' suggestions on her own big whiteboard: 15/100, 15/10 and 1 ½. Then the students were given time to pair-wise assess which of the suggested fractions were actually equal to 3/2. The teacher then randomly selected students to argue for why a certain fraction is equal to 3/2. The other students listened to the arguments and were then given the possibility to agree with the arguments or not, and to provide their own arguments or counter-arguments.

In the minute-by-minute formative assessment practice the teacher handled unpredictable situations and made decisions about teaching and learning in a matter of seconds. Even though the teacher had some thinking time between eliciting information and using information, unexpected questions or answers occurred which put the teacher in situations where flexibility and decisions were required instantly. Knowledge of how students learn mathematics was the most frequent type of teacher knowledge used during the activities and was for example used to understand different kinds of student misconceptions.

Final remarks

Together the studies show the feasibility of supporting teachers to develop their formative assessment practice in a way that improves student achievement. But, it can be expected that teachers would need substantial time and support.

There are different advantages with different ways of organising research. The description of the research program outlined in this article points to some of the benefits of a group working together and coordinating research endeavours. The results and experiences drawn from each of the above mentioned individual studies inform the design and understanding of the other studies. For example, experiences from the impact of the PDP on teachers' practice and their students' achievement, in combination with the study on the reasons for teachers' change and the in-depth analysis of the knowledge and skills used by one of the teachers' formative assessment practice, are currently used in a new combined research and school development project. A group of researchers with a common research agenda can more quickly and efficiently gather valuable experiences to be used in a specific context. In addition, the combined results from many related studies can provide a broader picture of a phenomenon under study. This may be a particularly valuable characteristic to be able to offer for a research group interested in engaging in collaboration with schools or municipalities.

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Short presentations

Students determining the median for different data sets: a spectrum of responses

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The present study reports on 359 students' responses to two test items on the median. Among the responses were other measures of location such as the arithmetic mean and the midrange. Moreover, among those who used a median strategy there was a spectrum of sources of confusion in the data set. In one test item the data were given in a table of signed integers and some students ignored the negative signs in the data. In the other test item the data were given in a bivariate diagram. Instead of correctly using the horizontal coordinate of the data, several students used the axis grading or the vertical coordinates as data. A conclusion is that the representation format of the data had a large effect on the achievement on the two test items.

Mot en förklaring av den Nya matematikens framgång och misslyckande i Sverige

Johan Prytz

Uppsala University

Den Nya matematiken infördes på bred front i den svenska grundskolan då Lgr69 trädde i kraft, vilket var relativt sent i jämförelse med andra länder. I Sverige hade under en stor del av 1960-talet gjorts försök med den Nya matematiken. Läromedel och styrdokuments hade testats och utvecklats genom försöksundervisning i ett flertal skolor, i vissa klasser upp till tre år. Resultaten från försöksundervisningen redovisades bland annat i rapporten Nordisk skolmatematik. Vid slutet av 1960-talet fanns det både i och utanför Sverige en tydlig kritik mot den Nya matematiken. I denna presentation redovisas en analys av denna kritik. I analysen prövas kritiken mot de resultat som erhölls från den svenska försöksundervisningen under 1960-talet.

Samhällets olycksbarn – en studie av elever med låga prestationer i matematik

Ingemar Karlsson

Lunds universitet

Syftet med studien som ingår i ett avhandlingsprojekt är att genom litteraturanalys av tidigare forskning samt intervjuer av lärare och elever redovisa förklaringar till uppkomsten av matematiksvårigheter. En begränsad forskning har medfört svårigheter att ge en enhetlig definition av begreppet dyskalkyli. Denna term definieras ofta som en biologiskt influerad avvikelse vilken kännetecknas av svårigheter att lära och tillämpa matematik. Sociokulturella faktorer, exempelvis föräldrarnas utbildning och kulturella kapital får istället allt större betydelse för elevernas resultat i skolan.

Framework of linguistic properties to compare mathematics tasks in different languages

Cris Edmonds-Wathen, Ewa Bergqvist & Magnus Österholm

Umeå University

This study aims to construct a framework of linguistic properties of mathematical tasks that can be used to compare versions of mathematics test tasks in different natural languages. The framework will be useful when trying to explain statistical differences between different language versions of mathematical tasks, for example, differences in item functioning (DIF) that are due to inherent properties of different languages. Earlier research suggests that different languages might have different inherent properties when it comes to expressing mathematics. We have begun with a list of linguistic properties for which there are indications that they might affect the difficulty of a task. We are conducting a structured literature review looking for evidence of connections between linguistic properties and difficulty. The framework should include information about each property including methods used to measure the property, empirical and/or theoretical connections to aspects of difficulty, and relevance for mathematical tasks.

Linguistic properties of PISA mathematics tasks in different languages

Ewa Bergqvist, Frithjof Theens & Magnus Österholm

Umeå University

The mathematics PISA tasks are primarily supposed to measure mathematical ability and not reading ability, so it is important to avoid unnecessary demands of reading ability in the tasks. Many readability formulas are using both word length and sentence length as indicators of text difficulty. In this study, we examine differences and similarities between English, German, and Swedish mathematics PISA tasks regarding word length and sentence length. We analyze 146 mathematics PISA tasks from 2000–2013, in English, German, and Swedish. For each task we create measures of mean word and sentence length. To analyze if there are any differences between the three language versions of the tasks, we use *t*-tests to compare the three languages pairwise. We found that in average, the German versions have the longest words, followed by Swedish and then English. Average sentence length was highest for English, followed by German and then Swedish.

Teaching mathematics as contribution to interaction

Andreas Eckert

Linnaeus University

The theoretical discourse on students learning has reached a state of maturity over the years whereas the theoretical discourse on teaching has fallen behind. The aim is to open up for a discussion about teaching in terms of contribution to interaction in the mathematics classroom. Symbolic interactionism forms the base of the theoretical discussion and learning is operationalized through the learning metaphor of contribution. The result is a conceptual framework of teachers engaging in social interaction and a negotiation of meaning, they actively contribute to the negotiation as well as they transform their own understanding of prior events. As meanings are derived from and handled in social interaction, it allows us to define the role of the teacher as an active contributor to students' development. By conceptualizing teachers as active contributors, we are one step closer to operationalize teaching within a theoretical frame of symbolic interactionism.

Using mathematical modelling activities to motivate biology students to learn mathematics

OLOV VIIRMAN, SIMON GOODCHILD, YURIY ROGOVCHENKO

University of Agder

This short presentation describes a collaborative project between two Norwegian centres of excellence in higher education in which mathematical modelling tasks are introduced to biology students as a means of motivating students to engage more deeply in mathematical studies. An ongoing mathematics teaching developmental pilot study is described, and some results regarding students' affective responses are presented – their motivation to engage in the tasks and in mathematics. The responses indicated that they found the activities worthwhile, and that similar activities would be of value as a part of their regular course in mathematics. These responses suggest that inclusion of mathematical modelling in authentic situations may have a positive impact on these students' motivation to study mathematics, and we conclude that these results support the continuation of the project on a larger scale.

Nationella prov i matematik: stöd vid betygssättning?

MARCUS SUNDHÄLL, FRIDA WETTERSTRAND, PER NILSSON & CHRISTIAN LUNDAHL

Örebro universitet

De svenska nationella proven har över tid haft olika syften. I samband med läroplansreformen 2011 tillkom syftet att kontrollera lärares bedömningar. I samband med att Örebro universitet fick i uppdrag av Skolverket att undersöka nationella provens inverkan på lärare och skolans arbete i grundskolans år 6 och 9 under hösten 2014 till hösten 2015 framkom en tydlig problematik kring det kontrollerande syftet, i synnerhet inom matematik. Resultatet av studien visar att lärare i matematik är, jämfört med lärare i andra ämnen, särskilt intresserade av att få sitt bedömningsarbete granskat. Däremot ser vi en tendens till att det fokus som finns, särskilt från media, på diskrepanser mellan provbetyget på nationella proven och slutbetyget har en negativ inverkan på lärares möjligheter att använda provbetyget som stödjande vid betygssättning.

Educational planning in mathematics as a part of macro-sociological structures

Helena Grundén

Linnaeus University.

All teachers in mathematics somehow plan for their teaching. They have considerations and make decisions that will influence what is happening in the classroom and thereby also what opportunities their students have to learn mathematics. Considerations and decisions are made in a social practice with power relations operating both within the practice itself and between practices. In a forthcoming study about planning of mathematics teaching these power relations will be explored. In this presentation different methods for exploring the power relations are discussed.

Conceptualizing mathematical reasoning – a literature review

Alexandra Hjelte

Örebro University

Is there a universal conceptualization of mathematical reasoning in mathematics education research? By investigating articles in the three highest ranked journals over the past ten years I have found a scattered picture of how mathematical reasoning is conceptualized. There is a need for a more systematic approach to understanding and analyzing mathematical reasoning.

Discourse analysis as a theory and tool investigating inclusion in mathematics

Helena Roos

Linnaeus University

In this paper initial thoughts of research methodology and theory in an upcoming Ph. D. project is presented. This project is an extension of a previous licentiate project regarding inclusion in mathematics from a teacher perspective.

Ett ramverk för att analysera matematiktexter med avseende på relationer mellan textens delar

Anneli Dyrvold

Umeå universitet

In order to understand more about difficulties related to the reading of mathematics text it is important to understand the role different features of the task text plays in the interpretation of the text. The proposed framework enables an analysis of particular textual features that make a text stick together, namely cohesive features. The framework is based on theory for cohesion and has been developed to catch important features of a mathematics text. Nine different types of cohesive relations are defined; these relations exist both within natural language, and between natural language and other semiotic resources. The framework has been developed to enable reliable coding of a substantial amount of text for the purpose of statistical analyses.

Mathematics teachers' communication about educational goals – a comparison between students' beliefs, teachers' descriptions and teaching

Lena Heikka

Luleå University of Technology

The aim of this study is to explore Swedish upper elementary school students' experiences of mathematics teachers' assessment practices, with a focus on educational goals communicated between the teacher and the class. In this multiple-case study with an ethnographic approach, three cases are viewed from a holistic perspective, by adapting *Visual model of the curriculum policy, design and enactment system* by Remillard and Heck (2014) to a Swedish context.

Results of the study show the complexity and variation of the communication about educational goals in relation to the syllabus in mathematics. Students in all three cases express and show a lack of knowledge of syllabus in mathematics and the textbook is considered as a concretization and visualization of the syllabus content. Teachers' expressed lack of knowledge about the syllabus in mathematics is probably due to insufficient implementation efforts of the curriculum, Lgr11.

The discourse regarding the multilingual student in need of support in test-instructions

Anette Bagger

Umeå University

This paper discusses parts of the discourse on multilingual pupils in need of support in the national test in mathematics in the third grade. A content analysis was done on the test-instructions from the years 2010–2014. A shift in the discourse was seen, from being about students in need to being about students with disabilities. The results show that instructions have moved from a relational towards a more categorical perspective on the student. One implication following this is that teachers receive less guidance in their mission to help pupils who need language support.

A mathematical representation of the heart

Djamshid Farahani

University of Gothenburg

The research reported here aims at providing possible explanations of how students interpret graphical representations of natural but complex phenomena. Mathematical representations such as diagrams, histograms, functions, graphs, tables and symbols normally make it easier for us to communicate and understand abstract mathematical concepts or other phenomena described in mathematical terms. Our investigation concerns upper secondary school student's understanding and interpretations of mathematical representations of a hearts work. The outcome further indicates that those students' alternative conceptions about graphical patterns and distance-time graph can be explained by ontological categorization of existing concept.