Students’ strategies to continue geometric number sequences

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Number sequences can be useful tools for teaching generalization, functions, or variables, for instance. Consequently, there are many studies that have studied students’ perception of number sequences and the strategies used to continue those sequences. However, a large part of the studies have been using arithmetic or quadratic number sequences. In this paper we present a study of students’ strategies to continue non-contextualized geometric number sequences. Interview data from 18 students in years 9 to 12 (age 15–19) (in Sweden) was analysed. Five qualitatively different strategies have been discerned in the data. These strategies are not completely overlapping the strategies previously described in literature.

Number sequences can be useful tools in mathematics education. Patterns and number sequences have been suggested to help students better understand the use of variables and to practice students’ ability to generalize (Mason, 1996; Orton & Orton, 1999). Horton (2000) has suggested that students could benefit from number sequences when learning about linear and exponential models. To continue a given number sequences, i.e. to find the next number in a sequence of the type (2, 4, 6, 8, _) is a part of a generalization process. However, continuing a sequence and express generality verbally is easier than to describe the same thing using algebraic notation (Zazkis & Liljedahl, 2002). Frobisher and Threlfall (1999) claim the importance of students meeting sequences of different types in the mathematics classrooms. Number sequences could be of different types and are classified according to structure and regularity.

In a *repeating sequence* a particular unit is repeated as in, e.g. (1, 2, 3, 1, 2, 3, 1, 2, 3, …) where the unit (1, 2, 3) is reoccurring.

In an *arithmetic sequence* an element can be found by adding a constant term to the preceding element. The sequence (2, 4, 6, 8, 10, …) is arithmetical since each pair of consecutive numbers are separated by a constant term (in this case 2).

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A *quadratic sequence* is a type of sequence where the difference between elements is in itself an arithmetic number sequence. An example could be (2, 4, 8, 14, 22, …). This sequence is quadratic since the difference between two consecutive number elements is comprising an arithmetic sequence, in this case (2, 4, 6, 8, …).

In a *geometric sequence* each pair of consecutive numbers have the same ratio. Or, if we put it differently, an element in a geometric number sequence can be found by multiplying the previous element by a fixed number. An example of a geometric number sequence is (2, 4, 8, 16, 32, …). This is geometric since each pair of consecutive numbers are separated by a constant factor.

In addition to the different types, number sequences are being presented to students in different representations: Word problems, Visual (pictorial), Table, Geometric and Numeric (number sequence) (Ye, 2005).

The strategies students use to continue number sequences have been described in different studies. Stacey (1989) described four methods of using number patterns to solve problems: *counting method, difference method, whole-object method* and *the linear method*. She studied students aged 9–13. Hargreaves et al (1999) also included younger students. Their students were 7–11 years old, and perhaps consequently Hargreaves et al (1999) also found slightly different categories of strategies: *looking for differences between terms, looking for the difference between the differences, looking for multiplication tables, looking at the nature of the numbers, looking at the nature of differences and combining terms to make other terms*. Bishop (2000) described six distinct strategies students aged 12–15 used to continue number sequences: *modell, multiply, apply proportional reasoning, skip count/add, use an expression,* and *other*. Ekdahl's (2012) study focused on the different ways number patterns were discerned by students (9–11 years old). Six different categories were identified. In summary these categories were associated with the way a part (a number) or several parts (numbers) in the number sequence were related to each other or to the whole sequence, alternative to an extension of the given sequence. Classroom studies of repeating patterns have been discussed by e.g. Papic (2007) and Warren & Cooper (2006). Arithmetic number sequences are probably the most frequently used types in mathematics education research and have been used in a large number of studies of students' strategies to continue number sequences (Bishop, 2000; Ekdahl, 2012; Hargreaves, Threlfall, Frobisher & Shorrocks-Taylor, 1999; Lin & Yang, 2004; Stacey, 1989). Students' strategies when continuing quadratic sequences have been described in several previous studies (Ekdahl, 2012; Hargreaves et al, 1999; Lin & Yang, 2005).

Basically all of the studies mentioned above involve increasing arithmetic number sequences. Some authors separate between linear and non-linear patterns. They then compare arithmetic number sequences with quadratic number sequences. However, how do we know that this should be the distinction – that quadratic and geometric sequences (both non-linear) are solved with the same strategies? Are the strategies researchers have found students using for arithmetic and quadratic sequences qualitatively different from strategies used to continue geometric sequences or any other type of number sequence? This is what we would like to explore. Hence, the aim of this study is to describe the qualitatively different categories of strategies students use to continue geometric number sequences.

Method

In order to be able to find as many strategies as possible, in terms of strategies to continue number sequences, a screening test was designed. Seven groups (classes) of students were screened with the test in order to look for candidates to interview. The groups were chosen from different levels and different schools; two groups (about 50 students in total) in lower secondary school, year 9 (age 15–16), and five groups (78 students) in upper secondary school (Science programme), years 10 to 12 (age 16–19).

The screening test comprised three number sequences, one arithmetic (3, 5, 7, 9, _), one quadratic (1, 4, 9, 16, _) and one geometric (1, 3, 9, 27, _). The students were asked to explain (in writing) their strategies to continue the different sequences. Based on the written answers to the tests, 18 students were selected for interviews (8 students from the secondary school groups and 10 students from the high school groups). The selection (which student to interview) was made on basis of the written screening test in order to embrace as large a variation as possible in students' strategies. A large variation is crucial in order to describe the different strategies students use (Marton & Booth, 1997). We are aware that the interview situation in itself can influence the students' answers (Hunting, 1997). We also know that the way tasks are designed can affect students' strategies when they generalize number patterns (Chua & Hoyles, 2013; Samson & Schäfer, 2007). Therefore, particular care was taken to ensure that the students were given the opportunity to describe their strategies in any way they felt suitable (written or verbally), and the interviews were semi-structured due to this consideration.

The tasks given to the students during the interview were non-contextualized, meaning that they were given just as numbers on a paper. In Ye's (2005) vocabulary we have given the students the problems only in the format of numeric number sequences – not visual (pictorial). The number sequences given to the students during the interview were (2, 4, 8, _), (1, 4, 16, _), (_, 125, 625, 3125) and (2, 8, 26, _), respectively. Three of the sequences are increasing with a traditional blank in the end, and one, (_, 125, 625, 3125) is in practice a decreasing number sequence written in increasing form (increasing numbers to the right). The purpose of using different kinds of number sequences, not only geometric sequences, was to be able to include as many different strategies as possible. The particular sequences the students should evaluate were tested in pilot interviews in order to include as many and as divergent strategies as possible. The sequences were handed to the students one-by-one, each on a single paper. Each number sequence in the interview was presented with numbers separated by blank spaces and with a line at the missing number the students were expected to find. The students were asked to continue the sequences and to find a general expression to describe the sequences. The interviews were audiotaped and later verbatim transcribed. Each interview took about 20–45 minutes.

We were inspired by phenomenography (Marton, 1981) and the method used in Ekdahl (2012). She focused on the different ways in which students discerned different number sequences. However, in our analysis we focus on the strategies to solve number sequences and search for similarities and differences between the students' strategies and descriptions. The strategies found were categorized in accordance with Marton & Booth (1997) and emerged from the different strategies the students used. A more detailed description of the data collection method has previously been given in two project theses (Lindahl & Tegnefur, 2012; Lindahl & Tegnefur, 2013).

Results and discussion

We found five qualitatively different categories of strategies for students to continue the geometric number sequences. There is no particular hierarchic order between the different categories. In brief, the different categories we have found are: *Operating with each number separately, Looking for common factors, Looking at the nature of the number, Looking at the difference between numbers* and *Looking at the element and its place (index).*

The categories comprise different strategies and can include a variation of different strategies, but with common features. The different strategies are described in detail below. Excerpts and examples from students are inserted in order to exemplify the strategies.

Operating with each number separately

What characterises strategies in this category is that it involves operations on a single particular number in order to find the next number in the sequence. These strategies appear to be focusing particularly on the numbers. Figure 1 shows an example where a specific operation (addition, multiplication) is applied for the entire sequence, but in order to find the next number the operation is applied only on the present number. In this example the student operates on the 2 to get a 4 and operates on a 4 to reach 8. Another example in the same category can be found in the immediate student response.

Interv.: Then we go for the next sequence [(1, 4, 16, _)].

Student: Then I would like to take 16 times 16 ... 32 [sic!], because ... Or, wait, it does not work. I thought 1 times 1, but it does not work because 4 times 4 is 16, but 1 times 1 is still not ... This was a bit more complicated ... 1 times ... Let's see if one can take 1 times 4. No, it does not work. Wait, yes, I would take 16 times 4, I guess ...

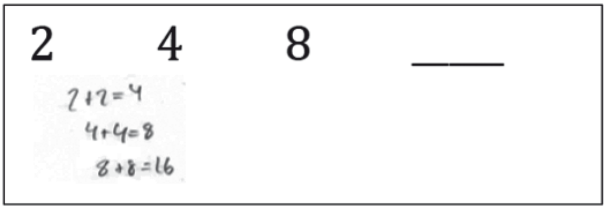

Figure 1. *In this case the student appears to operate with each number separately*

From the excerpt we see that the student's initial strategy is to operate with each number separately, in order to move from 16 to the next (unknown) number only 16 is operated on. One could argue that it is the square (4 x 4 = 16) that tricks the student to apply this strategy. However, as shown in figure 1 there are other situations too where strategies like this are tested.

These strategies are based on that the next number is sought by trying to apply a certain operation (to the sequence). But the operand(s) is limited to the present number. Hence, strategies in this category are based on operation with each number separately, not involving any other numbers or any difference between numbers.

Looking for common factors

Strategies in this category also focus on the numbers, but here a common factor is sought for in order to find the next number. The focus is not on each number separately, but on the collective property of the numbers in the sequence. In contrast to the previous category, here the multiplier (the factor) is the same for all numbers in the sequence. An example is shown in figure 2. Here the student shows the idea of looking for factors as the base in the powers.

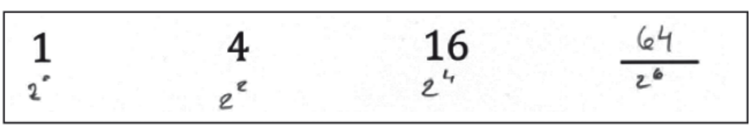

Figure 2. *The strategy seems in this case to be to look for a common base of the powers that constitute each number*

A related strategy is when the students for instance have shown that they look for factors to reach from one number to the next. As is shown in figure 3, the students could e.g. describe the operation on a number to find the next number. The operation, and particularly the operands, are more general, and not separate for each number. The strategies in this category do not include the multiplication

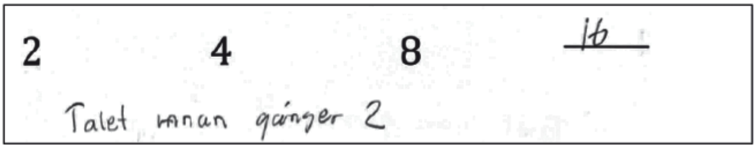

Figure 3. *This student strategy is described as "the number before times 2", which we interpret as if the student is looking at factors of the numbers in the sequence*

table. We consider looking for common factors to be qualitatively different from recognizing particular numbers from a multiplication table.

Both Hargreaves et al. (1999) and Stacey (1989) have described how students are looking for common features, like multiples. However, as they studied arithmetic and quadratic sequences and we study geometric sequences the common thing to look for must be different. In our case the students look for a common factor or a base in a power, whereas in their case the students can look for a common multiple. If we compare with the result by Ekdahl (2012) this category could be related to the perception of number sequences as related to equal motion between several parts of the sequence.

Looking at the nature of the number

Strategies in this category have in common that a particular nature of the numbers in the sequence is sought to find the next number. An example is the student that identifies a common property of the numbers in the sequence (_, 125, 625, 3125), as in this quote:

> It [the number] will probably end with 25, considering that it ends with 25 on all places, I would say.

In this category we can also find strategies based on trying to fit the numbers in the sequence into a particular multiplication table (although, not applicable to geometric sequences). A related example can be taken from one student trying to find a general formula for the sequence (2, 4, 8, _) expressed the following.

Student: There should be a 2 somewhere ...

Interv.: Why do you want a 2 here somewhere?

Student: Because 2 is twice as much ... Times 2 or power of 2 or something, maybe ...

A similar strategy can be seen in figure 4. The student has in this case sought for a formula including a number "2". In any case we consider the students efforts to be focusing on a common property or the collective nature of the numbers. We note that there is a wide span within this category. The properties of the number that is looked for can be of very different type. In the category *looking for the nature of the numbers* by Hargreaves et al (1999) they reported on properties such as odd and even numbers. In contrast, we observe other properties (here: that the numbers end with particular digits) being in focus when forming

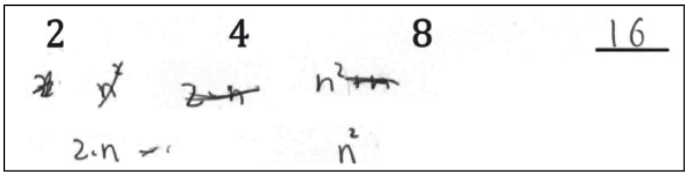

Figure 4. *The strategy seems in this case to look for known formulas all including a "2"; n^2 (written twice), $2+n$, n^2+n and $2 \times n$. The leftmost note could not be interpreted*

a strategy to find the next number. This could, however, be explained with that Hargreaves et al. (1999) studied younger students (age 7–11), whereas our study focuses on older students (age 15–19).

Looking at the difference between numbers

In this category the strategies are based on the fact that the difference between numbers is in focus. The difference is generalized in order to find the next number. An example is shown in figure 5. The strategy is explicitly stated as an operation with the differences between the numbers.

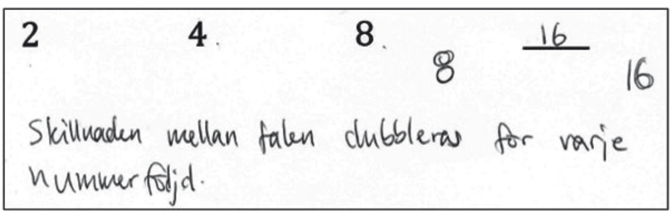

Figure 5. *This strategy is explicitly described as "the difference between the numbers is doubled for each number sequence"*

The strategy can also be found using powers, as in figure 6. In this example the student has shown the differences as repeated multiplication 2 x 1, 2 x 2, 2 x 2 x 2 and 2 x 2 x 2 x 2, respectively. We can compare this example with the strategy in figure 2. In that example the powers belong to the numbers, and in this example (figure 6) the powers (the repeated multiplication) belong solely to the difference between the numbers. As the given numbers are limited to three the sequences can be considered either as quadratic or as geometric. We observed that the sequence (2, 4, 8, _) gave particularly interesting data for the study. In this case there are two possible differences (4 – 2 = 2, and 8 – 4 = 4) and two possible ratios (4 / 2 = 2 and 8 / 4 = 2) to handle. Hence, if the strategy is to look at the difference we could anticipate either to look at the increasing difference (quadratic number sequence) or to look at the fixed ratio (geometric number sequence). This (quadratic/geometric duality) appears to be particularly evident in this number sequence. One student describes the duality as:

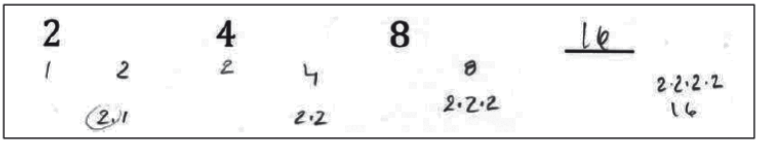

Figure 6. *In this example the student shows a strategy to find the generalization of the difference between the numbers*

Student: There it increases with 2 and then 4, then it should be 14

Interv.: Then it will be 14. Why should it be 14?

Student: Because here it increases with 2 and then ... or no, it can be both. It can be 16 too.

The operation on the difference can be of different types. Within this category we observe strategies where the differences are found through multiplication, addition or powers. However, we also observe a related strategy where the different differences are combined. In figure 7 is shown an example where the differences themselves are added.

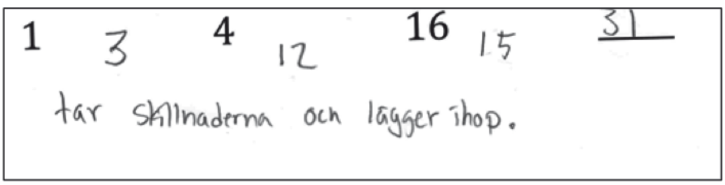

Figure 7. *The student has written "takes the differences and adds"*

Nevertheless, the common feature of the strategies in this category is that they are all based on an operation with what is not explicitly shown in the number sequences, but rather on what is in between what is shown – the differences between numbers. Both Ekdahl (2012) and Hargreaves et al (1999) describe the perception of what is between the numbers and points out the importance of the difference between numbers when generalizing number sequences.

Looking at the element and its place (index)

Strategies in this category look at the ordinal (index) of the element and its relation to the number at that place. A student shows an example of this strategy when trying to continue the sequence (2, 8, 26, _) by dividing 26 with its position or ordinal number (3). Similarly, another student tries to find a generalization by manipulating 16 (the third number) in the sequence (1, 4, 16, _) by operating on that number with 3 (number 16's ordinal number). Yet another student uses the strategy like this:

> Then I think one has to find something in common between these ... 2, 8, 26 ... and I am thinking that 8 is the same as 2 to the power of 3 and one

> should try to fit in some kind of order in the sequence too. Then it should be a one, a two, a three, a four, ... One cannot put *n* there either because if one changes 2, 8, 26 in these to a *n* it does not work either.

One could argue that the strategies based on looking at the element and its place is part of the category *Looking at the nature of the number*. However, in this case it is not really the inherent properties of the number that is in focus – it is the external property of its place in the number sequence that has a central role.

Discussion on the generality of strategies

Are the strategies students use different depending on whether we as researchers study geometric sequences or any other type of sequences? We note that the categories of strategies we have found have many similarities with, but do not completely map, the strategies by either Stacey (1989), Hargreaves et al (1999) or Bishop (2000). Particularly there are categories in Hargreaves et al's study on arithmetic and quadratic number sequences that are related to strategies we have identified. On the other hand, *Looking at the element and its place (index)* is not described in any of the studies on arithmetic sequences.

The representations we use for our number sequences are numerical which is different from the pictorial representation used by for instance Stacey (1989) and Bishop (2000). Actually, in many of the previous studies the patterns were given in visual (pictorial) representations. Possibly this could be a reason why we find slightly different strategies. Moreover, we do note that the students in our study is older than the students in previous studies. Mathematical experiences could play a role in the choice of strategy. However, we cannot completely rule out that the type of number sequence (arithmetic or geometric) can play a role in the difference we see in the strategies the students use compared to previous studies.

Conclusion

We have found five qualitatively different categories of strategies that students use to continue geometric number sequences. The categorized strategies have many similarities with strategies described in literature, but are not completely the same.

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