

Swedish students in upper secondary school solving algebraic tasks – What obstacles can be observed?

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To understand more about students' difficulties when doing algebraic problem solving Duval's semiotic theory and a mathematical modelling cycle are used to identify what obstacles can be observed. The results show that when the students have to perform transformations between two different semiotic representation systems – a conversion – the obstacles get visible.

The focus in this paper is to investigate what obstacles can be observed while students are doing algebraic problem solving, and how these obstacles can be characterized. The students are in the first year of the social science- and natural science program in three different upper secondary schools in Sweden. Algebra is a quite new area for the students at this level and it is known as an abstract and problematic area. A large body of research over a long period of time has documented the learners' difficulties both with the use of variables and the understanding of the nature of algebra as such (see e.g. Puig & Rojano, 2010; Stacey & Chick, 2010). Algebra is seen by many as a very abstract part of mathematics and as it is often taught, it has been characterized as "a watershed for most people" (Mason, 1996, p. 65).

However, algebra is versatile. When algebraic symbols are applied, it depends on the specific problem what one sees in them and what one is able to see. Drijvers (2003) suggests that problem solving in algebra is more difficult than problem solving in other areas. Among other things, this may depend on the abstract level at which algebraic problems are to be solved and the algebraic language with its specific symbols.

The use of letters in mathematics has been pointed out as a difficulty for students. In mathematics, letters are used in equations, in formulas, in functions, for generalization of pattern and so on (Drouhard & Teppo, 2010), and it is essential that students understand that letters always stand for numbers. It has been known for a long time that understanding the role of letters in algebra is very difficult for many students (see e.g. Küchemann, 1981; MacGregor & Stacey, 1997).

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The overall aim of this study is to learn more about upper secondary school students' difficulties while doing mathematics and how they interpret the mathematical content. For this particular paper I am going to address the following research question.

What obstacles can be observed when students in upper secondary school discuss algebraic problem solving in groups and how can these obstacles be characterized?

To answer the question a theoretical framework consisting of two parts will be applied. The identification of obstacles will be analysed using a modelling cycle (Lester & Kehle, 2003) to find where in the cycle the obstacles appear. To further characterize the obstacles, theory about transformations between semiotic representations (Duval, 2006) will be used.

Theoretical framework

Transformations between semiotic representations

Mathematical knowledge is a special kind of knowledge. It is not like other sciences because mathematical concepts or objects are abstract. Therefore there is no direct access to mathematical objects. To gain access to these objects the only way is by using semiotic representations. However, the signs have no meaning of their own and depending on whom you ask, you may get different answers, depending on the person's conceptions and experiences of the particular object (Duval, 2006). Duval proclaims that "the leading role of signs is not to stand for mathematical objects, but to *provide the capacity of substituting some signs for others*" (p. 106). This is what Duval refers to as *transformation* and he describes two different types of transformations, *treatments* and *conversions*. Treatments are transformations within one semiotic system, such as rephrasing a sentence or solving an equation. Conversion is a transformation that involves a change of semiotic system but maintaining the same conceptual reference, such as going from an algebraic to a graphic representation of e.g. a function. Duval uses the word *register* to denote a semiotic system that permits a transformation of representations (p. 111). Duval claims that changing representation register, i.e. performing a conversion, is the most challenging transformation for students.

Duval groups registers into *monofunctional* and *multifunctional*. A monofunctional register involves mathematical processes, which mostly take the form of algorithms (e.g. algebraic formulas). A multifunctional register consists of processes that cannot be made into algorithms (e.g. natural language) but involves other types of cognitive functions such as communication, awareness and imagination. Furthermore he distinguishes between *discursive* and

non-discursive registers where the former type is of the kind that e.g. involves statements of relations or properties, or statements about inference or computation, and the latter type consists of e.g. figures, graphs and diagrams. This gives four types of registers and transformations can take place between (conversions) and within (treatments) all four types (see Duval, 2006, p. 110).

In learning mathematics, the cognitive complexity of comprehension is touched through various kinds of conversions, more than through treatments. For example in a conversion task, when the roles of source register and target register are inverted, the problem can be changed for the students, and then they often fail. Many misunderstandings lie in the cognitive complexity of conversion and the change of representation. In a conversion a rephrasing can change the complexity of the situation. A conversion where the transformation from one register to another can be done by translating "sign by sign" turns out to be easier to handle than one where this is not the case. Duval refers to these types as *congruent* transformations and *non-congruent* transformations, respectively (2006, pp. 112–113).

The mathematical modelling cycle

Problem solving involves mathematical modelling. To solve a problem students have to first simplify the complex settings (Lester & Kehle, 2003). That involves interactive use of a variety of different mathematical representations.

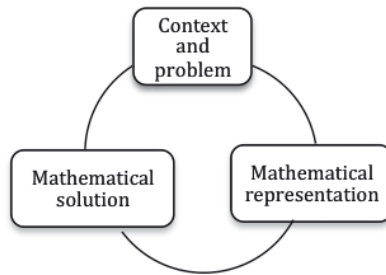


Figure 1. *A mathematical modelling cycle (Based on Lester & Kehle, 2003, p. 98)*

According to Lester and Kehle (2003), the problem solving process begins with a translation of the *problem* posed in terms of reality, into abstract mathematical terms. This involves making a decision about what could be omitted, how the key concepts are connected, and selecting mathematical concepts/variables. The next phase results in manipulation of the *mathematical representation* into a *mathematical solution*. Finally the solution has to be translated back into the terms of the original *problem*.

Method

Almost 100 students in the first year of upper secondary school at three different schools from a mid sized municipality in the middle of Sweden participated in the study. The three classes attended either the social science or the natural science program.

The main focus, which is to investigate students' communication and interaction regarding the mathematical content, places the study in an interpretative paradigm (Ernest, 1994). To grasp the students' interpretations of the communication, they were gathered in small groups (three to four in each group) and they had to solve a number of tasks. The tasks were chosen depending on what was discussed during the classroom observations, which preceded the group sessions. To create a situation where the students had to discuss and communicate the mathematical content, as well as to challenge them as a group to solve tasks they might have been unable to solve individually, the tasks had to have a somewhat greater level of difficulty than the students were accustomed to. To make it possible to study their interpretations, the external expressions of these interpretations will have to serve as material. That is to say, it is what students communicate, in words, actions, writing and gestures that make up the researchable data.

The problem solving situations with the student groups were video and audio recorded and some field notes were collected. Students' solutions from the algebraic problem solving were also collected to serve as a basis for the analysis.

The video and audio material was scrutinized several times and most of the material was transcribed verbatim for further analysis. In addition to oral communication, relevant non-verbal actions and interactions were included in the transcripts. The transcripts were scrutinized and categorized. In the first step the problem solving situations were divided into three different categories, one for each of the three transitions between the three boxes in the modelling cycle (see figure 1).

- 1 Problem → mathematical representation
- 2 Mathematical representation → solution
- 3 Solution → context and problem

Then each category was analysed regarding the students' discussion about the mathematical content. The results will be presented with examples from each category.

Student task

The students were given a number of algebra problem solving tasks that have been used in national tests in Sweden. All the tasks were algebraic problem solving tasks and the particular task presented below was chosen for this paper

because it was a little more difficult than the problem solving tasks in the text-book and it includes all the steps in the modelling cycle. The students have to interpret the context to understand the formula. After they have solved the task they have to interpret their answer and explain the formula using their own words.

When a freezer is turned off, the temperature inside rises. The following formula can be used to calculate the temperature (y) in degrees Celsius after the freezer has been turned off for x hours.

$$y = 0.2x - 18$$

- a Find the temperature inside the freezer if it has been turned off for two hours.
- b How long has the freezer been turned off if the temperature inside it is 0°C ?
- c Explain in your own words what the formula means.

(Skolverket, 2005, p. 4)

The algebraic formula $y = 0.2x - 18$ includes two variables, one independent (x) and one dependent (y). The formula represents a function because to every value of x there is exactly one value of y . In part a) what is needed is to replace x with 2 and calculate which temperature that corresponds to. In part b) one possible solution would be to construct an equation, $0 = 0.2x - 18$, and solve this for x . Another possibility would be to invert the original function and express x as a function of y . Then one could substitute the number 0 for y and get the corresponding value of x . The number 0.2 has the unit degrees/hour and that means that the temperature rises 0.2 degrees every hour. It is not explicitly said that the temperature is minus 18 degrees when the time is zero, so this is left to the students to interpret. Also the fact that 18 is subtracted could cause confusion because this must be interpreted as $y = 0.2x + (-18)$. Then the formula makes sense as expressing the final temperature as the sum of the increase in temperature after x hours and the initial temperature.

Students' solutions of the task

The results and analysis are based on the categories in the modelling cycle and will be presented with three short episodes from two different student groups' discussion of the task. The excerpts are selected based on their content and how the students treated the mathematical content. These are examples to show how the students reason.

Transition between problem and mathematical representation

The first and the third episode are taken from the same group, consisting of Lollo, Johan, Chris and Per. In the first episode they are in the process of translating the context to the mathematical representation.

Episode 1

The group has started with task a, discussing the meaning of x and y .

- 1:1 Lollo: y is degrees
1:2 Johan: What isn't y hours?
1:3 Chris: Thus y is degrees and x is hours
1:4 Johan: Yes ...
1:5 Lollo: The freezer has been turned off for x hours
1:6 Per: That is like one fifth
1:7 Johan: Yes one fifth, that is five, six minutes ... No twelve minutes.
1:8 Per: Yes that's right ... now I thought totally wrong.
1:9 Johan: Oh twelve minutes minus eighteen it is um ... minus ...

Lollo and Chris seem to understand the meaning of x and y [1:1, 1:3] but they do not interrupt Johan when he develops his interpretation of $0.2x$ as 0.2 hours, which he correctly calculates to be 12 minutes. However, it is not certain that Lollo and Chris have another interpretation of x than Johan and Per because Chris says that " x is hours" [1:3], and that does not necessarily mean "number of hours". Johan and Per seem to be convinced about their conclusion [1:7–1:8]. However, Johan gets some trouble when calculating 12 minutes minus 18 [1:9]. Here we may assume that he is not able to find a good interpretation of what his solution means in terms of the situation. Johan's interpretation of x as the unit "hours" ($0.2x \leftrightarrow 0.2$ hours) instead of $0.2x \leftrightarrow 0.2^\circ/h$ times the number of hours is not an unusual interpretation. This fits within the category "letter used as an object", as described already by Küchemann (1981, p. 104). The task here is about making a conversion from $0.2x$, in the discursive and monofunctional register, into "0.2 degrees per hour times the number of hours", in the discursive and multifunctional register. This is a non-congruent conversion, since the multiplication sign in $0.2x$ is invisible. However, Johan makes a congruent conversion when translating $0.2x$ "sign by sign" into "0.2 hours".

Transition between mathematical representation and solution

The most frequent solution procedure was the one shown at the beginning of episode 2. The students did not construct an ordinary equation to solve part b. Instead as a part of the solution, they used repeated addition and mental calculation

Episode 2

In this episode Karin, Fia and Sebbe are trying to solve the task.

- 2:1 Karin: How much did it go down every hour, was it zero point two? Couldn't you calculate so as to get one degree, how many hours that takes? Thus one whole degree and then take that eighteen times?

- 2:2 Fia: Um
2:3 Karin: Do you understand?
2:4 Sebbe: Yeah, wait, I have to think
2:5 Karin: The question is what one degree is ...
2:6 Fia: Five times zero point two, it becomes warmer. Thus zero point two times five ...
2:7 Sebbe: [Sebbe computes, using the calculator] oh, oh, oh
2:8 Fia: Yeah, but it feels like zero point two. How many, how many hours did it take for it to rise one degree
2:9 Sebbe: Don't you take ...
2:10 Karin: It is five, zero point two, zero point two, zero point two, zero point two and zero point two times ...
2:11 Fia: Yes, I think so
2:12 Karin: So it is
2:13 Sebbe: Don't we take eighteen divided by zero point two ... Yes it is.
[Sebbe takes the calculator and looks at the display]
2:14 Karin: Okay try that
2:15 Sebbe: It is ninety hours

The group reached the solution through discussion when they sort out the problem and Karin starts with the question "How much did it go down every hour, was it zero point two?" The temperature rises with 0.2 degrees per hour but most of the participating groups come to the conclusion that since the temperature approaches zero it goes down. Their interpretation that the temperature goes down could also have to do with the fact that 0.2 is less than 1, and that a connection is made to the fact that multiplication with a number less than 1 makes the result smaller. The formula, $y = 0.2x - 18$, is given in the monofunctional register but the interpretation is expressed in the multifunctional register (natural language). Karin [2:1] claims that the temperature goes down. Still, her procedure, to find out how many hours it takes to get one degree and then multiply that by 18, would give the correct answer. It is not clear that all students have the same interpretation of the situation. For example Fia [2:8] says that they should find "how many hours it takes for it to rise with one degree". However at the end Sebbe come up with another strategy [2:13–2:15] when he divides 18 with 0.2. In this discussion the students move back and forth between registers and although they find the correct solution it is not obvious that they have interpreted the formula correctly.

Explanation of the formula

The third episode involves the same group as episode 1. They have through a long discussion solved parts a) and b). In this episode they are trying to explain the formula (part c) of the task.

Episode 3

- 3:1 Per: Yes it is minus eighteen and zero point two and that is like degrees [points at the formula]
- 3:2 Johan: y is degrees
- 3:3 Per: zero point two x , is the number of hours
- 3:4 Chris: zero point two is the number of degrees it increases every hour
- 3:5 Per: Or decreases
- 3:6 Chris: Yes, no you can see that it increases there [points at $0,2x$ in the formula]
- 3:7 Per: Yes
- 3:8 Johan: So then, like ... So, that thing, which is before the x , zero point two is then ... the number ...? [writes at the same time as he talks]
- 3:9 Chris: Number of degrees since it is x every hour
- 3:10 Per: Yes but what is eighteen?
- 3:11 Johan: What, what is eighteen?
- 3:12 Chris: That is how much it was ... at the beginning
- 3:13 Per: Yes ... eighteen degrees
- 3:14 Chris: Minus eighteen degrees

It seems like Per still thinks that $0.2x$ is the number of hours [3:3] and he suggests that the temperature drops [3:5]. This interpretation of 0.2 is common in the data for this study. Many other groups did the same interpretation. Like in the second episode also this group seems to do the same interpretation. It is reasonable to interpret that this connects to the fact that if you multiply something with a positive number less than one, the result gets smaller, than the number you started with. However Chris points at the number 0.2 and explains that it is 0.2 that determines if it increases [3:6]. This seems to be a clarification for Johan too [3:8], that the number, 0.2 is not the number of hours, which Johan stated in the first episode and maintained throughout the whole solving process. Their dialogue in the last lines [3:10–3:14] shows that at least Per and Johan have not interpreted -18 as degrees. They may see it as just a subtraction, which was a common interpretation in other groups. They do not see it, as it is the temperature at the beginning.

Through the solving process and discussion the group is convinced. With help of Chris the group seems at the end to have interpreted the formula.

This is a conversion since the mathematical formula is in the monofunctional register and the explanation in natural language is in the multifunctional register. When the students interpret -18 as subtract 18 it is an example of a congruent conversion. They translate the formula word by word. They should rather interpret it as "add negative 18" in the formula.

Discussion

In this study the transition between different phases in the mathematical modelling cycle is analysed and the students' obstacles in each transition are characterized. To analyse the data Duval's framework of different semiotic registers is used.

The transition between problem and mathematical representation created some obstacles for the students. They have difficulties to interpret x as "the number of hours". Instead they interpret it as just the unit "hours". This is consistent with the phenomenon "letter as object" (Küchemann, 1981). However, seeing it as a conversion from the monofunctional to the multifunctional register, where the invisible multiplication sign in $0.2x$ makes the conversion non-congruent (Duval, 2006), may come closer to an explanation of why the phenomenon "letter as object" occurs.

In the transition between mathematical representation and solution the students mostly did not create an equation, they rather solved the problem with mental calculation and interpreted the formula term by term. They interpreted 0.2 as the temperature going down, perhaps because 0.2 is less than 1 or perhaps because the temperature approaches zero, which is taken to mean that it drops. However, most of the groups reached a solution after a while, without necessarily having interpreted the formula correctly. The obstacles in this transition occur also in the conversion, when they went between the two registers back and forth, when they should interpret that the temperature rises in the formula, $y = 0.2x - 18$.

The transition from the solution and back to the problem also causes obstacles for the students but is not shown in any episode because all interpretations of the solution were done during the solving process. Many of the groups have difficulties with the plausibility of the answer. They thought that it took too many hours for the temperature to rise.

The difficulties the students faced with explaining the formula were that they interpreted 0.2 as the temperature decreasing. Another obstacle was the meaning of -18. Many students seemed to see it just as a number to subtract. The obstacles occur even in this transition when the students have to change register, from the formula in the monofunctional to natural language in the multifunctional.

To summarize students' obstacles in their problem solving one can conclude that all obstacles that are shown in this study arise when students are forced to go between the registers, and most of the obstacles arise both in transition between problems and mathematical representation, and also between the solution and the interpretation of results.

The phenomena observed in this study have been observed before, but using the framework of Duval made it possible to shed new light on the phenomena, and this could be taken into account when helping students to overcome the obstacles.

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