Brackets and the Structure Sense

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Brackets are essential structure elements in mathematics. However, students have shown to have scattered understanding of the concept of brackets and how they are used in mathematical expressions. In this paper we present data that illustrate students' perceptions of the word "brackets" and how these perceptions influence their use of brackets in numerical expressions. Based on our data we argue that the teaching of the concept of brackets also need to describe brackets as ordered pairs where each symbol has a unique counterpart and that insertion of brackets can, but does not have to, modify the structure of an expression.

Introduction

Students' understanding and misunderstanding of letters in and the structure of algebraic expressions has since long been well described (Küchemann, 1978; Rosnick, 1981; Kieran, 1989). A central set of symbols for the algebraic structure is the brackets. Brackets constitute an essential part of algebra and distinguish, together with rules for the order of operations, the algebraic language from spoken everyday language (Freudenthal, 1973, p. 305). However, students' understanding of the bracket symbols is not equally well documented in mathematics education research.

Typically the concept of brackets is taught alongside with rules for the order of operations. Brackets and their properties are often introduced to students in a single sentence saying that "brackets show what should be calculated first". However, this is not necessary always true. Two examples; in the expression $4 + 5 - (2 + 3)$ one could very well add 4 and 5 before adding the 2 and 3 within brackets, and when solving the equation $(x + 3) \cdot 2 = 8$ the first operation is not to calculate what is inside the bracket but to divide the equation by 2.

In addition, there are misconceptions of the word and the concept of brackets, some known and described in literature. It has been shown that students can interpret "brackets should be calculated first" as "brackets should appear first" in a left-to-right meaning (Kieran, 1979). In addition, brackets can, when used as a marker for negative numbers as is common in the Swedish mathematics teaching tradition, cause confusion to what should be calculated first (Kilhamn, 2012). As an example, what should be calculated first in (-2) – a negative two?

Moreover, Hewitt (2005) has shown that he word “brackets” often is translated literally into a mathematical expression – ignoring the structure of the expression to be written. He also showed that the word “bracket” appears ambiguous, as seen when students read out equations loud or translate text to equations.

Another possible cause of problem is that brackets are used with different purposes in mathematical expressions. Brackets can be used to emphasise the intended order of operation but otherwise be mathematically useless, like in $\frac{1}{(x+1)}$, or brackets can be necessary parts of the expression which without them would have another meaning, like in $2 \cdot (4+3)$. Linchevski and Livneh (1999) have suggested to use emphasising brackets in $a \pm b \cdot c$ type of expressions in order to detach the number (b) from the operation ($\pm$), supporting the learning of a structure sense. Useless, emphasising, brackets can indeed help students see algebraic structure (Hoch & Dreyfus, 2004), and emphasising brackets can increase success rates in arithmetic expressions (Marchini & Papadopoulos, 2011). But one has to be careful when using emphasising brackets as it has been shown that they may impede the learning of precedence rules (Gunnarsson, Hernell & Sönnerhed, 2012). Overall, there are plenty of reasons to look deeper into the teaching and learning of bracket symbols.

Aim and scope of the study

The aim of the study discussed in this paper is to analyse students’ perception of mathematical brackets. We would like to achieve this by answering the following research question: How do students perceive the word “bracket” and the concept of brackets in mathematical expressions?

Description of the study

For this study 84 students, aged 14-15 (school year 8), in eight different classes in four different Swedish schools participated in a paper-and-pencil questionnaire. The questionnaire contained ten tasks each including one or more expressions to evaluate. Each student was asked to evaluate in total 35 different arithmetic expressions, a few of them will be discussed in this paper. Details of the full questionnaire can be found in (Karlsson, 2011). No calculators were allowed during the test.

In the Swedish teaching tradition students typically first meet brackets in the seventh grade. By involving eighth grade students we therefore probed the students’ perceptions and their use of brackets in their initial phases of learning the concept, but they should have met brackets in their mathematics teaching at least the year before. The schools and classes were not selected by any statistical method, but had a reasonable distribution regarding gender, ethnicity and social background.

The data was analysed mainly by qualitative methods. However, to some degree the data were also quantitatively summarised. Though the main analysis was made by categorising the different perceptions that became evident in the students' answers. The focus in this brief report is not on the analysis of single students' different answers, but on describing the different sets of misunderstandings that came up in this study. The answers to the different tasks were therefore analysed (by categorisation) and the perceptions found were cross-correlated between different tasks. Hence, the categorisation system is not in focus in this paper, but rather the outcome of the cross-correlation between different tasks.

Results

The students were asked to choose what they perceived as a bracket. The actual question that was asked was "Which, or which one of the following is example of a bracket?" with the alternatives "(", ")", "( )" and "(3)" and with tick-boxes for each type. Figure 1 shows a Venn-like diagram of the distribution of student answers to this question. The numbers in the diagram in Fig. 1 show the number of students ticking each separate box. The majority of the students considered the empty pair of brackets and the brackets with a content to be exam-

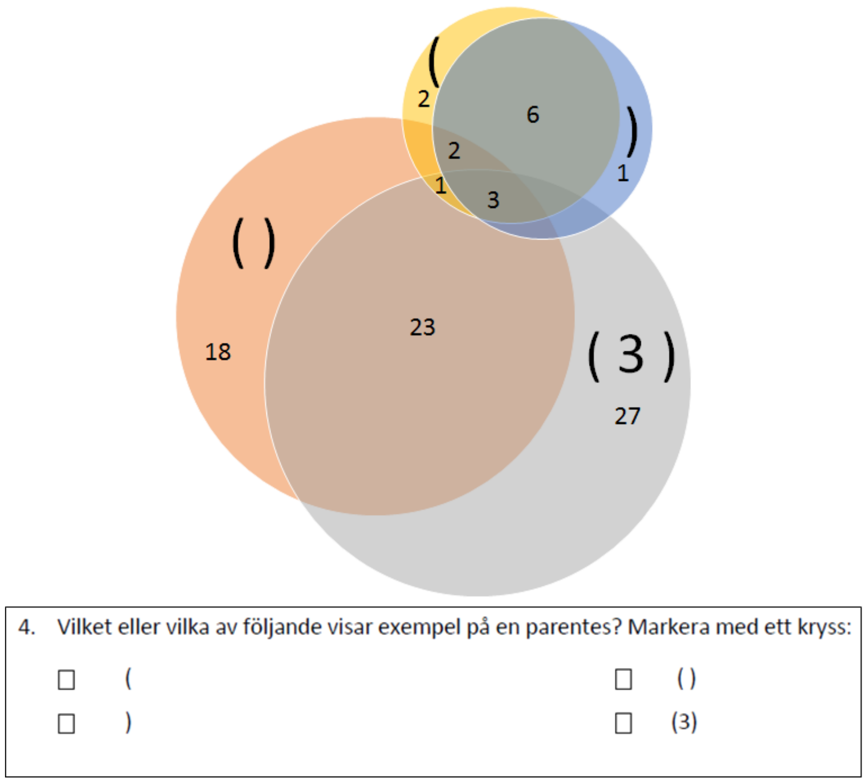


Figure 1: The space of all answers to the question "Which, or which one, of the following shows example of a bracket?"

ples of brackets. The largest group (27 students) marked only the alternative "(3)", a bracket with content, as an example of a bracket, whereas the second largest group (23 students) marked both "( )" and "(3)", but not a single left "(" or right ")" symbol. A small number of students (2+6+1, i.e. in total 9 students) answered that a single symbol, either a left-handed or a right-handed symbol or both – but not in combination, represent a bracket. An even smaller number of students (1+2+3, in total 6) marked both single symbols and symbols in combinations to be examples of brackets.

Figure 2 shows a few students' answers to which brackets that are considered unnecessary in a number of different expressions. In Figure 2(a) the question with the complete set of mathematical expressions is shown. The student has in this case marked all unnecessary brackets except the ones that emphasise the precedence of multiplication over addition/subtraction. Two examples of answers where the marked bracket symbols are not corresponding to a conventional pair are shown in Fig. 2(b)-(c).

In Figure 2(d) only a single bracket symbol is marked as unnecessary. The closing bracket in the midst of the expression appears to be considered as necessary. In the answer in Figure 2(e) it appears as if the student considers multiple brackets unnecessary, i.e. that it should be sufficient with a single bracket symbol. Almost the same kind of perception of brackets is shown in the answer shown in Figure 2(f), where outer multiple brackets have been deemed unnecessary.

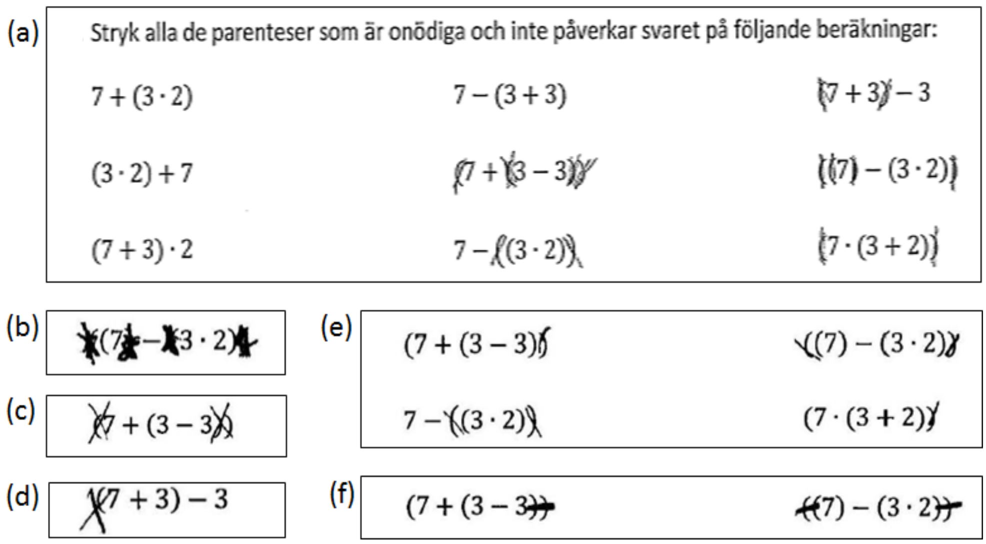


Figure 2: A selection of answers to the task "Cross out all the brackets that are unnecessary and do not affect the answer to the following calculation".

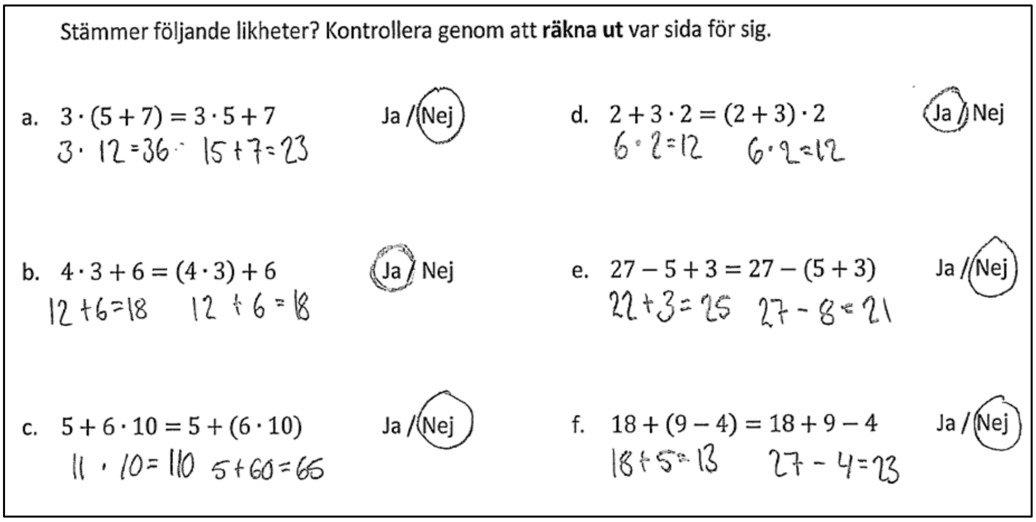
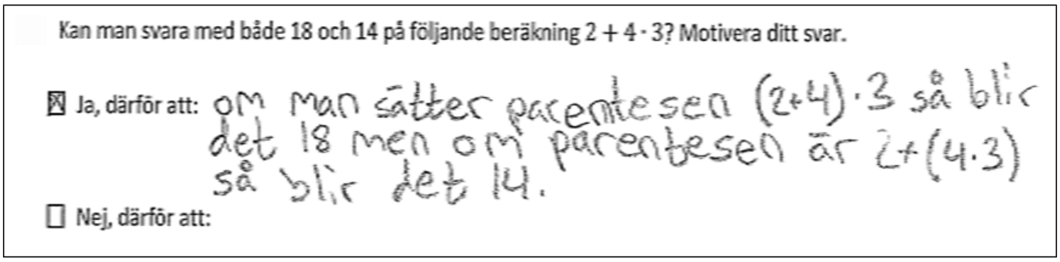


Figure 3: One student's answer to the question "Are the following equalities correct? (Yes/No) Verify by **calculating** each side separately".

The students were also asked to evaluate numerical equations, see Figure 3. In this task all six equations contained brackets on the left or the right hand side. In three expressions the brackets were mathematically useless and in the other three the brackets were necessary in order to maintain the structure of the expression. In the student answer shown in Figure 3 it appears as if the student regards brackets to signal precedence, but that without brackets the expression should be evaluated from left to right. Consequently the student answers that, e.g., $5 + 6 \cdot 10$ should be evaluated differently than $5 + (6 \cdot 10)$, and that $(2 + 3) \cdot 2 = 6 \cdot 2$ [*sic*!] is the same as $2 + 3 \cdot 2$.

Another student answers the question whether the expression $2 + 4 \cdot 3$ is ambiguous, with "[yes, because:] if you put the bracket $(2 + 4) \cdot 3$ it will be 18 but if the bracket is $2 + (4 \cdot 3)$ it will be 14", see Figure 4. Hence, this particular student has answered that the evaluation of the expression depends on where you put the brackets. But this is not a single student phenomenon. A frequent answer to this question on the questionnaire was "yes" (23 students). However, among the other answers there were 5 blanks/don't know and a small number of motivations like "[no, because:] there could only be one answer". The student answer shown in Figure 4 is one of those revealing a perception of brackets as if they could be used arbitrarily. But also in the "no"-responses there were indications of alternative perceptions of brackets as in e.g. the answer "[no, because:] there are no brackets and then it must be 18". This latter can be seen as yet another example of when the absence of brackets leads to a left-to-right calculation.


Kan man svara med både 18 och 14 på följande beräkning 2 + 4 · 3? Motivera ditt svar.

☒ Ja, därför att: om man sätter parentesen (2+4)·3 så blir det 18 men om parentesen är 2+(4·3) så blir det 14.

☐ Nej, därför att:

Figure 4: One student's answer to the question "Is it possible to answer both 18 and 14 to the following calculation $2 + 4 \cdot 3$? Motivate your answer. (Yes, because:/No, because:)".

Discussion

Even though the students have been introduced to brackets there is still a wide spectrum of misconceptions that can be seen in the data. We cannot exclude, actually we find it very likely, that students' preconception of the word bracket plays a major role to this. Possibly, everyday communication where single bracket symbols are frequently used as, e.g., in "smileys" :-) could have an influence on the perception of brackets as a single left or right arch. We also note that the Swedish language is ambiguous regarding the use of the word "parentes". In the official Swedish language the word *parentes* refers to an inserted expression ("inskjutet uttryck") according to the Swedish Academy glossary (The Swedish Academy, 2006), and a single bracket should be called "parentestecken". The equivalents of "opening bracket" and "closing bracket" are used but are often called "start parentes" and "slut parentes".

In addition, the phrase *within brackets* (note "bracket" in plural) would in Swedish be translated to *inom parentes* (singular). Hence, we anticipate that the students' language could be a source for some misconceptions observed in our data. One could argue therefore that this is a local problem, but as we see that problems regarding students' ways of handling brackets in mathematical expressions appears also in the English language (Kieran, 1979; Hewitt, 2005) we believe there are more general implications.

Students' perception of the word "brackets"

The alternatives the student could choose from in the question in Figure 1 were fixed and no openings for alternatives were offered. The options to mark were single left symbol "(", single right symbol ")", an empty pair of symbols "( )" and a pair of symbols with some content "(3)". Other alternatives could be possible, but we believe that the perceptions of brackets are mainly revealed in *how* the brackets are used in mathematical expressions, described in next section.

It is interesting to note, however, that only a small number of students do consider both single symbols and paired symbols to be examples of brackets. The majority considers a bracket to have a content (or possibly that it *can* have a content). This group represent 27 (+23), as shown in Figure 1. This is consistent with the viewpoint that in the Swedish language the word brackets ("*en parentes"*) represents an inserted expression, i.e. the content within a pair of brackets.

Students' perception of the concept of brackets

The different ways of perceiving brackets, as single symbols, as empty pairs or as pairs with contents can lead to problems when translating text to an algebraic expression, as shown by Hewitt (2005). In the full questionnaire (but not shown in this paper) we also included a similar task, and we find the word "bracket" interpreted as *single symbols* or as *empty pairs* or *pairs with contents* – the same categories as in the perception of the word brackets discussed above. However, we also find, in agreement with Hewitt (2005), that students do not consider the structure of an expression when translating words to symbols. A substantial part of the students does not seem to consider the structural properties of bracket symbols.

Even when the brackets are perceived as a pair they do not necessarily have to be perceived as ordered pairs. In Figure 2(b-c) we find examples where it appears as if the students are forming the bracket pairs somewhat arbitrarily. If we recreate the answer in Figure 2(b) the student have left the rounded brackets and crossed out the square brackets in this expression $[(7] - [3 \cdot 2)]$. Of course in this expression all brackets can be considered superfluous. The student does seem to acknowledge that brackets appears in pairs. But we focus on the new pair that the student forms. This is not a pair in the sense that a particular opening bracket has a corresponding closing bracket. What this student seems to have missed is that brackets appear in *ordered* pairs.

The same question also revealed another misconception. In Figure 2(e) a student answer is shown where multiple bracket symbols appear to be perceived as unnecessary. The student seems to consider $(7 \cdot (3 + 2))$ to be the same as $(7 \cdot (3 + 2)$. In this case the two opening brackets are considered to share the same closing bracket. It appears as if the student has missed that for every opening bracket there exists one *unique* closing bracket, and vice versa. Possibly this could also be true for the student giving the answer in Figure 2(f). But that answer could also be related to the answer in Figure 2(d). In this case it appears as if the brackets are perceived as only separating inner parts of an expression. The student in this example suggests that after removing the unnecessary brackets in the expression $(7 + 3) - 3$, what should be left is $7 + 3) - 3$. We believe this is an example of where brackets are considered to be single symbols, not pairs.

In conclusion our data suggest that brackets are perceived as single symbols, empty pairs of pairs with content. The pairs can by students be perceived as being formed by any two combinations of single bracket symbols and need to even be perceived as an even number of single symbols (e.g. when two "left brackets" are paired with one "right bracket").

Students' use of brackets as part of mathematical notation

Our data support the observation by Kieran (1979) that brackets are a signal of what should be calculated first. Even though it is not reported here we do also see examples in our data where students move brackets to the left to "do them first". However, the data also show examples of when the lack of brackets is taken as a signal that the rules for the order of operations do not apply. In Figure 3 we see such an example where brackets are used as necessary parts of the structure of the expression. But this example also reveals that in the lack of brackets the structure is considered different, i.e. left-to-right instead of precedence. The student seems to have missed that *brackets show the structure* of an expression. This appears also to be true for the student whose answer is shown in Figure 4. This student seems to have missed the information that *brackets cannot be inserted arbitrarily* without changing the structure of the expression (and the result of the calculation). We believe this shows that it has to be made clear that there is a close connection between the structure of a mathematical expression and where in the expression brackets can be inserted without distorting it.

In conclusion, we find that students do not necessarily perceive brackets as the important structure element described by Freudenthal (1973, p. 305). Brackets can be perceived as a signal to use the precedence rules, but without the brackets the expressions could be evaluated left-to-right. Brackets can be perceived as something that can arbitrarily be inserted into an expression.

Structure sense, brackets and educational implications

The term *structure sense* was coined by Linchevski and Livneh (1999) in order to describe difficulties in algebra based on lack of understanding of structure of arithmetic expressions. We believe that in order to fully understand the mathematical structure it is necessary to also, or possibly first, understand how terms are grouped and how different operations work together. But grouping of terms cannot be made arbitrarily. Hence, when teaching mathematical rules for the order of operations, emphasising brackets can be used. This is analogous to the use of emphasising brackets by Hoch and Dreyfus (2004) and Marchini and Papadopoulos (2011).

However, as supported by our data the present introduction of brackets appears to be insufficient. We therefore believe that the introduction of brackets needs to emphasise the properties of brackets, not just their place in the rules for the order of operations. Particularly, based on our data, we suggest that brackets are presented as ordered pairs where each bracket symbol has a unique counterpart. That the insertion of brackets is shown to be able to change the structure of an expression, but that brackets not necessarily have to induce such a change.

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