

Conceptualizing mathematical reasoning – a literature review

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Is there a universal conceptualization of mathematical reasoning in mathematics education research? By investigating articles in the three highest ranked journals over the past ten years I have found a scattered picture of how mathematical reasoning is conceptualized. There is a need for a more systematic approach to understanding and analyzing mathematical reasoning.

Introduction

The current review is part of an ongoing doctoral project, with the overall focus on theorizing mathematical reasoning and how digital learning environment can be designed to foster mathematical reasoning. This interest is motivated by an increasing awareness that reasoning is important to the learning of mathematics (Yackel & Hanna, 2003). In the curriculum for the Swedish school system mathematical reasoning is formulated as an ability that all students need to develop (2011).

The purpose of this paper is to describe how mathematical reasoning is conceptualized in mathematics education research. The primary focus is to investigate if there is consensus, differences and contradictions in how mathematical reasoning is conceptualized in mathematics education research.

Method

The review is based on articles from the three highest ranked mathematics education research journals in the web of science: *Journal for Research in Mathematics Education*, *Mathematical Thinking and Learning* and *Educational Studies in Mathematics*. Basing the review on articles from these three high ranked journals was assumed to strengthening the reliability of the review. I looked through the titles of the articles that had been published over the past ten years for the word reasoning. I wanted to see, when the writers placed the word reasoning in the title, if they also gave an explanation of how they understood the concept of reasoning. If the article mentioned the word reasoning I looked at if this was connected to particularly students' reasoning. If the article was about students at the university, reasoning in textbooks or teachers reasoning it was excluded. This scanning process resulted in 26 articles, presented in Appendix 1.

Result – Different explanations or theoretical backgrounds for the concept of reasoning

Out of the 26 articles, two did not explain how they conceptualize or understood reasoning. 8 articles gave unclear explanations on how they conceptualize reasoning.

In 16 articles the writers gave an explanation or a theoretical background for the concept of reasoning. They explicitly explained the concept of mathematical reasoning with a definition or criteria for how it can be identified.

From the 16 articles articulating an explanation of mathematical reasoning we gain a scattered picture of the meaning of mathematical reasoning.

Two main categories were identified: (1) Mathematical reasoning conceptualized in terms of justification, explanation and generalization. (2) Mathematical reasoning conceptualized as solution strategies that could be divided into subgroups within a mathematical content. One of the 16 articles did not fit to any of these two categories. In this article Lithner's model of imitative and creative reasoning was used to conceptualize and analyze reasoning.

In the first category there were different suggestions on how to understand and relate reasoning, justification, explanation, argumentation and generalization to each other. In four articles mathematical reasoning was defined or explained but they lacked a deeper, reinforcing and theoretical framework for the concept. In two articles informal inferential reasoning was used as theory to conceptualize reasoning. In two articles mathematical reasoning was explained within a semiotic theory. The last article had a theoretical framework that was developed that defined reasoning from inferences.

In the second category reasoning was always connected to a specific mathematical topic, for example geometry, algebra or proportionality. Reasoning was then explained as different solution strategies for this specific topic or an expression for different understandings within the topic, but there is no clear consensus between the articles on how this is categorized.

Discussion and implications

Of the 26 articles in this overview 21 articles did not have a theoretical framework that was used to give a base for the conceptualization of mathematical reasoning. Sometimes reasoning was not explained at all and a lot of the times reasoning was explained or defined with references to prior research but without being based in a theoretical framework. To enhance the scientific quality and the legitimacy of the research that focuses on mathematical reasoning it can be necessary to search for alternatives to find this theoretical framework. A start could be to look deeper into the theoretical frameworks that was used in some of the articles.

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Appendix 1

Table of the articles in the overview

Number	Authors	Title	Journal	Category/explanation of reasoning
1	Barrett, Jeffrey E. Clements, Douglas H. Klanderman, David Pennisi, Sarah-Jean Polaki, Mokaeane V.	Students' Coordination of Geometric Reasoning and Measuring Strategies on an Fixed Perimeter Task: Developing Mathematical Understanding of Linear Measurement	Journal for Research in Mathematics Education	Justification, explanation
2	Ben-Zvi, Dani Bakker, Arthur Makar, Katie	Learning to reason from samples	Educational Studies in Mathematics	Unspecific explanation of reasoning
3	Bishop, Jessica Pierson Lamb, Lisa L. Philipp, Randolph A. Whitacre, Ian Schappelle, Bonnie P.	Using order to reason about negative numbers: the case of Violet	Educational Studies in Mathematics	Solution strategies (including generalization)
4	Bishop, Jessica Piersson Lamb, Lisa L. Philipp, Randolph A. Whitacre, Ian Schappelle, Bonnie P. Lewis, Melinda L.	Obstacles and Affordances for Interger Reasoning: An Analysis of Children's Thinking and the History of Mathematics	Journal for Research in Mathematics Education	Solution strategies (including explanation)
5	Bjuland, Raymond Luiza Cestari, Maria Borgersen, Hans Erik	The Interplay Between Gesture and Discourse as Mediating Devices in Collaborative Mathematical Reasoning: A Multimodal Approach	Mathematical Thinking and Learning	Argumentation, generalization (semiotic)
6	Boesen, Jesper Lithner, Johan Palm, Torulf	The relation between types of assessment tasks and the mathematical reasoning students use	Educational Studies in Mathematics	Lithner
7	Brousseau, Guy Gibel, Patrick	Didactical Handling of Students' Reasoning Processes in Problem Solving Situations	Educational Studies in Mathematics	Justification
8	Chen, Chia-Ling Herbst, Patricio	The interplay among gestures, discourse, and diagrams in students' geometrical reasoning	Educational Studies in Mathematics	Explanation, argumentation (semiotic)
9	Dominguez, Higinio LópezLeiva, Carlos A. Khisty, Lena Licón	Relational engagement: Proportional reasoning with bilingual Latino/a students	Educational Studies in Mathematics	Gives no explanation of reasoning
10	Ellis, Amy B.	Connection Between Generalizing and Justifying: Students' Reasoning with Linear Relationships	Journal for Research in Mathematics Education	Justification, generalization
11	Garfield, Joan Le, Laura	Developing students' reasoning about samples and sampling	Educational Studies in	Unspecific explanation of

	Zieffler, Andrew Ben-Zvi, Dani	variability as a path to expert statistical thinking	Mathematics	reasoning (talks about generalization)
12	Gil, Einat Ben-Zvi, Dani	Explanations and Context in the Emergence of Students' Informal Inferential Reasoning	Mathematical Thinking and Learning	Justification, explanation, argumentation, generalization
13	Jacobs, Victoria R. Franke, Megan Loef Carpenter, Thomas P. Levi, Linda Battey, Dan	Professional Development Focused on Children's Algebraic Reasoning in Elementary School	Journal for Research in Mathematics Education	Solution strategies (including generalization)
14	Johnson, Heather Lynn	Secondary Students' Quantification of Ratio and Rate: A Framework for Reasoning about Change in Covarying Quantities	Mathematical Thinking and Learning	Unspecific explanation of reasoning
15	Jurdak, Murad Eid El Mouhayar, Rabih Raif	Trends in the development of student level of reasoning in pattern generalization tasks across grade level	Educational Studies in Mathematics	Solution strategies (including generalization)
16	Lannin, John K.	Generalization and Justification: The Challenge of Introducing Algebraic Reasoning Through Patterning Activities	Mathematical Thinking and Learning	Justification, argumentation, generalization
17	Lobato, Joanne Hohensee, Charles Rhodehamel, Bohdan Diamond, Jaime	Using Student Reasoning to Inform the Development of Conceptual Learning Goals: The Case of Quadratic Functions	Mathematical Thinking and Learning	Unspecific explanation of reasoning
18	Makar, Katie	Young children's explorations of average through informal inferential reasoning	Educational Studies in Mathematics	Unspecific explanation of reasoning (talks about generalization)
19	Makar, Katie Bakker, Arthur Ben-Zvi, Dani	The Reasoning Behind Informal Statistical Inference	Mathematical Thinking and Learning	Justification, explanation
20	Meletioui-Mavrotheris, Maria Paparistodemou, Efi	Developing students' reasoning about samples and sampling in the context of informal inferences	Educational Studies in Mathematics	Unspecific explanation of reasoning
21	Miyakawa, Takeshi Winsløw, Carl	Didactical designs for students' proportional reasoning: an "open approach" lesson and a "fundamental situation"	Educational Studies in Mathematics	Unspecific explanation of reasoning
22	Modestou, Modestina Gagatsis, Athanasios	Cognitive and Metacognitive Aspects of Proportional Reasoning	Mathematical Thinking and Learning	Solution strategies
23	Pfannkuch, Maxine Arnold, Pip Wild, Chris J.	What I see is not quite the way it really is: students' emergent reasoning about sampling variability	Educational Studies in Mathematics	Unspecific explanation of reasoning (talks about generalization)
24	Pittalis, Marios Christou,	Types of reasoning in 3D geometry thinking and their	Educational Studies in	Solution strategies

	Constantinos	relation with spatial ability	Mathematics	
25	Rubel, Laurie H.	Middle School and High School Students' Probabilistic Reasoning in Coin Tasks	Journal for Research in Mathematics Education	Justification, explanation
26	Walkington, Candace Petrosino, Anthony Sherman, Milan	Supporting Algebraic Reasoning through Personalized Story Scenarios: How Situational Understanding Mediates Performance	Mathematical Thinking and Learning	Gives no explanation of reasoning